$\sqrt{a^5}$; $\dfrac{4^{1/2}}{4^{1/3}} = 4^{1/2-1}$

$^{1/2} = \sqrt{25} \times \sqrt[3]{}$

ALGEBRA

FOR THE PRACTICAL WORKER

J. E. Thompson

MATHEMATICS LIBRARY FOR PRACTICAL WORKERS

PREFACE TO FOURTH EDITION

The popular demand for this book and for the other four in the MATHEMATICS LIBRARY FOR PRACTICAL WORKERS has been maintained through three editions and for over fifty years. The publisher hopes that this Fourth Edition will prove as valuable to a new generation of practical workers who must work with mathematics on a regular, practical, non-theoretical basis.

PREFACE TO THIRD EDITION

ALTHOUGH the approach to algebra used in this volume is unchanged from that used so successfully in earlier editions, every effort has been made to introduce certain new *concepts*, in keeping with contemporary developments in the subject matter. Moreover, there is added here a new chapter devoted to Boolean and other algebras. The student will find that an acquaintance with these will enable him better to understand some of their more common applications—notably, in computers—and it is hoped that these algebras will also prove profitable to the student in his study of advanced mathematics, particularly those areas of mathematics—such as vector calculus—that play such a prominent part in the natural sciences and engineering.

PREFACE TO FIRST AND SECOND EDITIONS

In olden days mathematics was regarded as a part of magic and at first only priests and medicine men were allowed to use it. Later it was a part of the stock in trade of astrologers and alchemists, and even today a person who is good in mathematics in school is often called by his friends a "wizard." But knowledge of the ancient science of mathematics is no longer a possession of the selected few. All of us are free to study any part of the subject; nearly all of us use mathematics in some form or other, and those of us who engage in engineering or similar pursuits find the calculus, trigonometry and other higher branches of mathematics among the essential tools of our trade. So also the practical man must have a knowledge of the basic principles of mathematics in order that he may fully understand their application in his work. Mathematics has become vital to industry and to business and must be a part of the education of everyone.

Appreciating that there are many who wish to carry their study of mathematics beyond the limits of their schooling, this series has been designed to present the various branches of fundamental mathematics so that they may be understandable without the assistance of a teacher. Such treatment should also be of interest to those who wish to review their knowledge of special branches and to refresh their acquaintance with methods of application by the study of a book that gives both the underlying principles and their use in business and in industry. The author, in his experience as an engineer and as a teacher of mathematics to students and laymen in evening schools and in colleges, has heard many of these men express the desire for books that may be used for home study, and the series of books to which this volume belongs is the result of an attempt to meet their needs and wishes.

The complete series consists of volumes on arithmetic, algebra, geometry, trigonometry, and calculus. The present volume on algebra, while not the first of the series in order of reading sequence, is, perhaps, the key book of the series in that it forms the reader's real

introduction to the principles of abstract mathematics, assuming only a knowledge of arithmetic.

The meaning of algebra and its position among the various branches of the general subject of mathematics are described more fully in the introduction to this volume and the reader is urged to familiarize himself with that section as a starting point for mastering the principles of the science. It will be noted that the book is divided into five parts, each of which treats of a single natural division of the whole subject or of a group of related shorter subjects. This method has been found useful in introducing the subject in that it makes a gradual and natural transition from the concrete and specific ideas of arithmetic to the more general ideas of algebra, particularly to the use of letters to represent quantities and to the fundamental methods of reckoning with letters instead of numerals. With this transition once made, the titles of the remaining parts are self-explanatory.

The methods of approach and presentation of each part have been made as direct, as natural, and as informal as possible. However, the book is scientific in its approach. It attempts to explain and not to side-step obstacles and difficult points. There can be no non-mathematical book on mathematics, and popularization may only be gained through clarity of expression and through a human and common-sense approach with a minimum of formality. It is realized that algebra is a somewhat abstract subject and the aim in writing this book has been to produce explanations which can be read and understood by the average adult reader who has a knowledge of arithmetic, without the need of a teacher or helper.

Reports and comments from many readers of the first edition of the book indicate that the plan has been successful, but more exercises and problems have been called for and requests have been made for the addition of certain subjects and more complete explanations of a few others. Such changes and additions have been made in this new second edition, but they are not large or numerous, and the general plan, scope and methods remain the same.

J. E. Thompson

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INTRODUCTION

In beginning the study of algebra, the stubent or reader frequently has the notion that he is entering a new field, a distinct subject which is entirely different from arithmetic. Such is not the case, however. Algebra is simply an extension and completion of arithmetic and the two really form one large subject; the study of numbers and their forms, combinations and uses. This complete subject is divided into the two parts called *Arithmetic* and *Algebra* simply for convenience in study and instruction, and because the viewpoint from which the science of numbers is considered is slightly different in the two parts.

In arithmetic the only numbers considered are the numbers used in counting, called whole numbers or *integers*, and parts or "pieces" of those numbers, called *fractions*. These fractions may be written in the form called common fractions, which occurs naturally in measurement, or the form called decimal fractions, which is more convenient for writing and in addition, multiplication, etc. When these numbers are once defined and described, the remainder of the subject of arithmetic has to do with the rules for their simple combinations (addition, subtraction, multiplication, division, square and cube roots) and their use in measurement, commerce and everyday affairs. Little attention is given to the basic reasons underlying the rules and the connections between the various rules, or to the extension of the rules and the wide general principles of which those considered are only a small and special part.

In algebra, the attention is not restricted to integers and fractions and their combinations and uses. In fact, the numbers used are not even of a special form and letters are used rather than the ten special symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, and these letters, or literal symbols, may represent *any numbers* whatever. Also, in arithmetic, all numbers are thought of as beginning at zero and running upward, *increasing*, and no thought is given to the question of what there may be on the other side of zero. Then, when we have to deal with, say, temperatures, which may be less than the arbitrarily chosen zero; or debts, in which the assets are less than nothing; or equal distances in opposite

directions which are obviously different but for which the same numbers must be used; for such cases as these we have no suitable numbers but have to describe the situation by means of cumbersome word descriptions, while in all ordinary cases we simply write the number with the symbol for degrees (°), dollars ($), feet (ft.), etc., and need say no more. We shall see that in algebra a new kind of number is used in such cases.

Again, in arithmetic we can add together any two numbers, large or small, but in subtraction we are only allowed to subtract the smaller from the larger and are forbidden to think of subtracting the larger from the smaller. And yet we know that when the temperature is, say, only 15° it can drop 20°; a man can be required to pay out $50 when he only has $25, etc. Thus, arithmetic makes addition work both ways but leaves subtraction a one-sided affair when we know that subtraction must also work both ways. In algebra this restriction is removed and the blank in our number system is filled up. There is also another restriction in arithmetic (in taking square roots) which is removed in algebra; we shall come to this later.

Besides enlarging and extending the number system of ordinary arithmetic, algebra also develops simple methods of writing numbers, a sort of shorthand which allows one symbol to represent any number and greatly simplifies the rules and methods for writing out addition, subtraction, multiplication, etc. When these rules are simplified and systematized the relations between them are pointed out and explained and general principles are then developed which include them all and extend them to new forms of calculation, by means of which problems are solved which cannot be solved by arithmetic alone, and other problems are solved much more quickly and easily than by arithmetic. Besides these extensions of arithmetic, many special kinds of operations are developed in algebra which are the very foundation of the higher branches of mathematics used in astronomy, physics, chemistry, engineering and other branches of pure and applied science.

It is easy to see now that algebra is not so much concerned with the *uses* or *applications* of numbers in calculation and solving problems, but in studying and extending their *meaning*, the *rules* of their operations, their theory or *principles* and the forms and *methods of writing* them. From this viewpoint, it may be said that algebra is the completion and fulfillment of the ideas which are only stated and illustrated in arithmetic, and is the alphabet, language and grammar, by means of which

mathematical thought and theories are expressed. To use a homely illustration from everyday life, we first learn to speak mathematics in arithmetic, while in algebra we learn its alphabet, its dictionary and grammar, and its written language. With this education we are then prepared to take up the study of the special branches of mathematics and their use in solving the problems of the various branches of science and engineering.

In this book, considerable emphasis will be laid on the symbols and methods of algebra ("alphabet, language and grammar") as the basis for such subjects as geometry, trigonometry, calculus, etc., and less attention will be paid to problem solving, only such everyday problems being considered as cannot be solved by simple arithmetic or are much more simply and easily solved by algebra than by simple arithmetic.

To any one interested in the spirit and methods of mathematics algebra is a subject of fascinating interest, and to any one who intends to study engineering, science, or higher mathematics it is an absolutely necessary preparation. The aim of this book is to bring out and cultivate this interest for the general reader and to prepare the way to further study for the serious student who may not be so situated as to have the advantage of formal instruction in organized classes.

We begin in Part I with the *fundamental methods* of algebra in which we get the new view of the familiar numbers and operations of arithmetic.* When these have been studied we will be in a position to use these methods in studying the general *fundamental principles* and extending them to develop the new methods which go beyond arithmetic. In the first few chapters, we shall "make haste slowly" until the fundamental methods are well understood. After that, we will be in a position to proceed more rapidly and the explanations can be made shorter and more concise.

* *Note.*—If the reader's knowledge of arithmetic needs to be refreshed or extended, reference may be made to the author's book "Arithmetic for the Practical Man," published by D. Van Nostrand Company, Princeton, N. J.

Part 1

FUNDAMENTAL METHODS

Chapter 1

SYMBOLS AND NUMBERS OF ALGEBRA

1. Algebraic Symbols. It will be recalled from arithmetic that in the chapters on mensuration the number 3.14159 . . . is of very frequent use in connection with circles. In order to avoid the frequent writing and speaking of this number, which is there necessary, it will be remembered that a single letter or symbol is used to represent it. This symbol is π (pronounced "pi"). When this number is needed in actual multiplication, division, etc., it is of course written out to the desired degree of accuracy, but in simply writing out rules, problems, etc., it is sufficient to write or refer to the symbol π.

If the diameter of a circle is known, the circumference can be found by multiplying the diameter by the number π. Thus, we say that, "the circumference equals π times the diameter." Using the usual sign ($\times$) to represent multiplication, and the sign ($=$) to indicate equality, this rule can be written as follows:

$$\text{Circumference} = \pi \times (\text{Diameter}).$$

This rule can be greatly abbreviated if we write the initial letters C and D instead of the words "circumference" and "diameter," giving

$$C = \pi \times D,$$

and this form is just as well understood as the full statement and much more easily remembered and *written*.

In the same way, the rule for the area of a rectangle, "the area equals length multiplied by width," or

$$\text{Area} = (\text{Length}) \times (\text{Width}),$$

can be very concisely written as

$$A = L \times W$$

if we let A, L, W represent the area, length and width, respectively.

We might go on to recite many more such examples in which this "mathematical shorthand," so to speak, makes the writing of rules

and problems in arithmetic much simpler, but the cases cited are sufficient to illustrate our meaning.

The letters used above to represent numbers, lengths, areas, etc., are much used in algebra and are called *algebraic symbols*. Obviously, if a letter or other symbol is to stand for a number, since numbers may be added, subtracted, multiplied, etc., we can also represent the addition, subtraction, etc., of the number symbols. Thus, we have already written

$$C = \pi \times D, \quad A = L \times W,$$

and if the sum of any two specified numbers is equal to a third number and these are represented by a, b, c, respectively, we can also write

$$a + b = c.$$

And again, since c represents the number which is the sum of the numbers represented by a and b, then a is the difference between c and b and we can write

$$c - b = a.$$

Again, suppose we are told that the sum of two numbers is 9 and that their difference is 5, and are asked to find the numbers. After a few trials of different numbers we find that the correct ones are 7 and 2. In ordinary arithmetic it would require two or three full lines to write out the problem but if we let x represent the larger of the unknown numbers and y the smaller, we can state the whole problem at once by simply writing

$$x + y = 9, \quad x - y = 5,$$

and when the symbols are once understood, these simple statements place the entire problem before the mind at one glance.

From the several illustrations given, we see that algebraic symbols may represent either *known* or *unknown* numbers.

2. Arithmetical Operations with Algebraic Symbols. We have just seen that numbers represented by algebraic symbols can be added, subtracted, multiplied the same as ordinary numbers. Therefore, we can write out statements involving unknown numbers and indicate their use in problems, adding, multiplying, etc., as well as if we knew their values, simply by using algebraic symbols.

As an example, suppose we are given the following problem: A boy makes a purchase which takes half the money he has. He then earns

twice as much as he had at first and finds that he has five dollars. How much money did he have at first?

Even though we do not know how much he had at first we can let it be represented by an algebraic symbol, say the letter x. If then x is the number of dollars which he had, what he spent was $\frac{1}{2}x$ and what he had left was also $\frac{1}{2}x$. The amount he earned, being twice what he had at first, is $2x$. This added to what he had left after his purchase amounts to five dollars. Therefore,

$$\tfrac{1}{2}x + 2x = 5.$$

This simple symbolic statement is the statement of the entire problem. In order to solve the problem we must find the value of the unknown number x. To do this we add the $\frac{1}{2}x$ and the $2x$ and obtain $2\frac{1}{2}x$. The symbolic statement then is

$$2\tfrac{1}{2}x = 5.$$

From this, of course, it must be that $x = 2$. That is, the boy had two dollars at first.

This simple problem and its algebraic solution show how we can treat a number symbol just as we might treat the number itself. Thus, we subtracted $\frac{1}{2}x$ from x and the remainder was $\frac{1}{2}x$; to $\frac{1}{2}x$ we added $2x$ and the sum was $2\frac{1}{2}x$. We may think of $2x$ as meaning the same as x multiplied by 2, or $2 \times x$, and we may think of the $\frac{1}{2}x$ as x multiplied by $\frac{1}{2}$ or divided by 2. These operations give us a partial idea of the manner in which symbols are used for numbers in algebra.

Let us now consider another operation with algebraic symbols. In the same way that we can multiply one number by another in arithmetic, as 2×3, or by itself, as 3×3; and as we can multiply a symbol by a number, as $2 \times x$; so in algebra we can multiply *any* two numbers or symbols. Thus, $2 \times x$, $a \times b$, $a \times a$, etc., and also $a \div b$, $2x \div y$, etc. When any number is multiplied by itself, as 3×3, $a \times a$, it is said to be *squared*, from the fact that the area of a square is found by multiplying the length of the side by itself. Thus, if a square is 3 inches on a side its area is $3 \times 3 = 9$ square inches and 9 is called the *square* of 3. Similarly, $a \times a$ is called the square of the number a. Since in squaring a number it is used twice as a factor the square of a number is also called the *second power* of the number. Thus, 3×3 is the second power of 3, and $a \times a$ is the second power of a. Similarly, $5 \times 5 \times 5$ is the third power of 5, and $a \times a \times a$ is

the third power of a. Since the volume of a cube is the third power of the length of its edge, the third power of a number is called the *cube* of the number. Thus, the volume of a cube which is 5 inches on the edge is $5 \times 5 \times 5 = 125$ cubic inches, and 125 is the cube of 5.

In the same way, the product obtained by using a certain number four times as a factor is the fourth power of the number. Thus, $b \times b \times b \times b$ is the fourth power of the number b, or, as it is abbreviated, "b to the fourth," or simply "b fourth." Similarly, $c \times c \times c \times c \times c$ is the fifth power of c, "c fifth." The product obtained by using any particular number any number of times as a factor is the corresponding power of that particular number, and the number is said to be *raised* to that power.

In indicating the powers of numbers it is inconvenient to write out the full product always. Thus, to raise 2 to the 6th power we have

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64.$$

For simplicity, this is written $2^6 = 64$, the figure indicating the power being a small figure written just above and to the right of the number which is raised to that power. It must be remembered that 2^6 does *not* mean 2×6, but $2 \times 2 \times 2 \times \ldots$, and so on 6 times (the string of dots meaning "and so on"). Similarly, 5^3 does not mean 5×3 or 3×5, but $5 \times 5 \times 5$, 5 used 3 times as a factor.

Using this method we would write,

$$5^3 = 125$$

$$3^2 = 9$$

$$a \times a = a^2$$

$$c \times c \times c \times c \times c = c^5$$

$$b \times b \times b \times b = b^4$$

$$2^6 = 64, \text{ etc.}$$

Of course, if we do not know what numbers a, b, c, etc., represent, we cannot tell what a^2, b^4, c^5, etc., are equal to, but we can still indicate by the symbols and the usual signs just *how* these numbers are multiplied, added, etc.

3. Exponent and Coefficient. The figure indicating the power to which a number or algebraic symbol is raised is called the *exponent* of the power, or, sometimes, of the number. In the case of 5^3, the 3 is the exponent of the third power of 5, or simply the exponent of the 5.

Similarly, in x^7, the 7 is the exponent of the seventh power of x, or the exponent of the x.

If an algebraic symbol or number is multiplied by another number, as $2 \times a$ or $2a$, the multiplier is called the *coefficient* of the symbol. Thus, here the 2 is the coefficient of the a. The coefficient, unlike an exponent, is a full-sized figure written to the left of and on the same line with the algebraic number or symbol.

After a number is raised to any power, as 5^3, the result can then be multiplied by any other number. Thus, 4×5^3 is the same as 4×125 which is 500. This is written $4 \times 5^3 = 500$. Similarly, 4 times a cube is $4 \times a^3$, or, as it is usually written in algebra, $4a^3$. Here the 3 is the exponent of a and the 4 is the coefficient of a^3. Similarly in ay^4, which means a times the fourth power of y, 4 is the exponent of y and a is the coefficient of y^4.

4. A New Kind of Number. Although we speak here of a new *kind* of number it will be seen that we really have reference to a new way of *representing* a familiar kind of number. Suppose the temperature of the atmosphere at a certain spot as registered on a thermometer is 10° and a 20° drop takes place, what does the thermometer then register? We say 10° below zero. But the very fact that we add the words "below zero" shows that this 10° is not the same as the original 10°. Also, although we cannot in the ordinary sense subtract 20° from the original 10°, the temperature has actually decreased 20°. We, therefore, need some way to represent the result of subtracting the 20° from 10°, and also to distinguish the new thermometer reading of 10° from the original reading of 10°.

Both these things are accomplished by using the familiar plus (+) and minus (−) signs of arithmetic with the meanings somewhat extended. Thus, although we know that in the ordinary sense 20 cannot be subtracted from 10 we do know that when the temperature is 10° it can decrease 20°. This is indicated by writing

$$10 - 20 = -10.$$

Here the first minus sign has its usual meaning of *indicated* (even if not *actual*) subtraction, and the second minus sign indicates that the new reading of 10° *below* zero is of the *opposite kind* from that of the original reading, which was 10° *above* zero. The original reading, above zero, is referred to as +10° and is called *positive*, and the new reading (−10°) is called *negative*.

Numbers which indicate the opposite sense or meaning from that of ordinary numbers in any connection are called *negative numbers*, and the ordinary numbers are called *positive numbers*. A height of 500 feet above standard sea level would be $+500$ feet (positive) and a depth of 500 feet below sea level would be -500 feet (negative). A credit of \$75 would be $+75$ dollars and a debit of \$75 would be -75 dollars. If a distance measured forward, to the right, or upward from a chosen reference point is taken as positive, then a distance measured backward, to the left, or downward is negative.

By using this scheme, it is immediately obvious that a great simplification is introduced in writing numbers. Thus, instead of writing "10° above zero" or "10° below zero" we write, simply, $+10°$ or $-10°$, respectively. Instead of writing "500 feet above sea level," or "500 feet below sea level" we write, simply, $+500$ feet or -500 feet, after the reference or zero point is chosen. In science and engineering this distinction is of the greatest usefulness and where mathematical calculations are to be made it is an absolute necessity, although, in everyday life and very simple calculations we can still get along after a fashion by using cumbersome descriptive words to explain our meanings, just as it is possible to go through life without being able to read or write.

It is part of the business of algebra to develop and explain the method of writing and calculating with negative numbers so that when they are met in the sciences and higher mathematics they may be used as readily as ordinary (positive) numbers.

5. Combining Positive and Negative Numbers. We have given as one definition of negative numbers the statement that they are the *opposite* of ordinary numbers, which are called positive. How are we now to understand the operation indicated by

$$10 - 20 = -10?$$

The meaning of this operation may be understood by taking another thermometer example. If an *increase* of temperature of 20° should occur when the reading is already $+10°$ the thermometer would then read $+30°$ and this would be indicated by writing

$$+10 + 20 = +30,$$

so that an *increase* is a positive change. Therefore a *decrease* must be a *negative* change, since it is a change of the opposite kind from an in-

crease. If then the 20° decrease occurs when the reading is $+10°$ then we can only say that the expression

$$+10 - 20 = -10$$

means: -20 *combined* with $+10$ gives -10 as a result, and not that 20 is *subtracted* from 10. The same word may be used in the case of $+10 + 20 = +30$. Thus $+20$ combined with $+10$ will give a larger number of the same kind, namely, $+30$. On this view, since $+10$ and -20 are opposite kinds the combination of the two will give a result in which the $+10$ cancels off -10 of the -20, so to speak, leaving a -10 of the original -20. In the same way, if the reading is $+10$ and a 10 degree decrease occurs, that is, a $-10°$ change, the result is zero and this is indicated by writing

$$+10 - 10 = 0.$$

We can think of this as meaning that a positive 10 is combined with a negative 10 to give a zero result, or as -10 added to $+10$ to give a zero sum, or as 10 subtracted from 10 to give a zero remainder.

The same result would be obtained if a 10 degree rise should occur when the reading is -10. This operation would be indicated by

$$-10 + 10 = 0.$$

This may again be thought of as combining two numbers of equal amount but opposite kinds, or as adding a positive number to a negative number of the same magnitude. In the same way, if the reading is -10 and a 20 degree rise occurs the new reading is $+10$. That is,

$$-10 + 20 = +10.$$

Similarly, we obtain by operating with such numbers, purely by algebra and without regard to their physical significance:

$$1 - 2 = -1$$

$$3 - 3 = 0$$

$$-3 + 3 = 0$$

$$4 + 4 = 8$$

$$-7 - 3 = -10$$

$$-3 - 7 = -10$$

$$+3 + 7 = +10$$

$$15 - 20 = -5, \text{ etc.,}$$

the signs (+) and (−) indicating *combination* of like or unlike numbers rather than specific addition or subtraction. We shall see later, however, that the old meanings of these signs are not discarded when the new meanings are introduced.

If a number is written in the ordinary manner without any sign it is to be understood to be positive (+).

6. Multiplication and Division of Positive and Negative Numbers. In order to derive the rules for the multiplication and division of signed numbers, we shall consider the motion of a train which travels at the constant rate of 40 miles per hour. We shall label the starting point of the train 0. We shall consider motion to the right of the starting point as positive (+) and motion to the left of the starting point as negative (−). We shall agree that future time will be regarded as positive (+) and past time as negative (−). The diagram below indicates the position of the train three hours in the future (point A) and two hours in the past (point B).

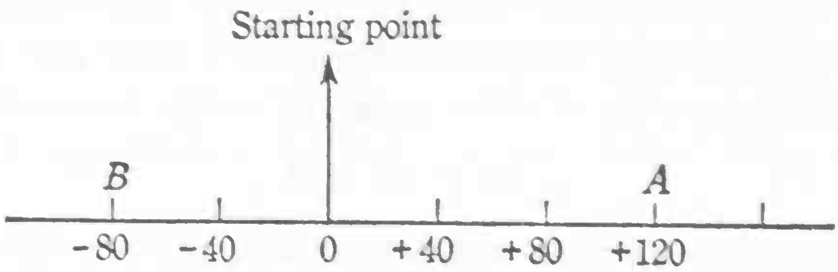


FIG. 1.

Next, let us consider the following four problems and their solutions by the use of signed numbers.

1. If the train is now at the starting point and travels to the right where will it be 5 hours from now?

$$\text{Motion to the right} \times \text{Future time}$$
$$(+40) \times (+5) = +200$$

It is clear that the train will be 200 miles to the right, or at +200.

2. If the train is now at the starting point and travels to the left, where will it be 5 hours from now?

$$\text{Motion to the left} \times \text{Future time}$$
$$(-40) \times (+5) = -200$$

It is clear that the train will be 200 miles to the left, or at −200.

3. If the train is now at the starting point and has been traveling to the right, where was it 5 hours ago?

$$\begin{array}{ccccc} \text{Motion to the right} & \times & \text{Past time} & & \\ (+40) & \times & (-5) & = & -200 \end{array}$$

It is clear that the train was 200 miles to the left, or at -200.

4. If the train is now at the starting point and has been traveling to the left, where was it 5 hours ago?

$$\begin{array}{ccccc} \text{Motion to the left} & \times & \text{Past time} & & \\ (-40) & \times & (-5) & = & +200 \end{array}$$

It is clear that the train was 200 miles to the right, or at $+200$.

Let us collect our results compactly and examine them.

$$(+40) \times (+5) = +200$$
$$(-40) \times (+5) = -200$$
$$(+40) \times (-5) = -200$$
$$(-40) \times (-5) = +200$$

These results may be summarized briefly in the following simple rule.

RULE: *The product of any two numbers of the same sign is positive while the product of two numbers of opposite signs is negative.*

Since the above results could have been obtained with *any* other numbers as well as 40 and 5, they hold good with algebraic or literal (letter) numbers. Therefore, we can write the results as follows:

$$(+a) \times (+b) = +ab$$
$$(-a) \times (+b) = -ab$$
$$(+a) \times (-b) = -ab$$
$$(-a) \times (-b) = +ab$$

Since division is the inverse of multiplication and since in the examples given above

$$(+40) \times (+5) = +200, \text{ then } \frac{+200}{+5} = +40$$

$$(-40) \times (+5) = -200, \text{ then } \frac{-200}{+5} = -40$$

$$(+40) \times (-5) = -200, \text{ then } \frac{-200}{-5} = +40$$

$$(-40) \times (-5) = +200, \text{ then } \frac{+200}{-5} = -40$$

These results may be summarized briefly in the following simple rule.

RULE: *The quotient of any two numbers of the same sign is positive while the quotient of two numbers of opposite signs is negative.*

Since the above results could have been obtained with *any* other numbers as well as 200 and 5, they hold good with algebraic or literal (letter) numbers. Therefore, we can write the results as follows:

$$\frac{+a}{+b} = +\left(\frac{a}{b}\right)$$

$$\frac{-a}{-b} = +\left(\frac{a}{b}\right)$$

$$\frac{+a}{-b} = -\left(\frac{a}{b}\right)$$

$$\frac{-a}{+b} = -\left(\frac{a}{b}\right)$$

7. Signs Used in Algebra. In algebra, the signs plus (+) and minus (−) have their ordinary meaning, *indicating* addition and subtraction (whether they can actually be performed or not), and also serve to distinguish between opposite kinds of numbers, positive (+) and negative (−). In such an operation as $+10 - 10 = 0$, as explained before, the minus sign means that the minus 10 is combined with the plus 10 to give a zero result or that 10 is subtracted from 10 to give a zero remainder. Similarly, we may also say that $10 - 20 = -10$ means the combination of $+10$ and -20 to give a result -10, or that 20 is *algebraically subtracted* from 10 to give the *algebraic remainder* -10.

The so called "double sign" (±), which is read "plus-or-minus," is sometimes used. It means that the number or symbol which it precedes may be "either plus or minus" or "both plus and minus."

As in arithmetic, the equality sign (=) means "equals" or "is equal to."

The multiplication sign (×) has the same meaning as in arithmetic.

In many cases, however, it is omitted. Thus, ab means the same as $a \times b$ and $2 \times c$ means the same as $2c$, twice c. We cannot write 23, however, for 2×3, as 23 has another meaning, namely, the number twenty-three. Therefore, in general, the multiplication sign ($\times$) may be omitted between algebraic symbols or between an algebraic symbol and an ordinary arithmetical number, but not between two arithmetical numbers. A dot ($\cdot$) placed between any two numbers and a little above the line (to distinguish it from a decimal point) is sometimes used as a sign of multiplication. Thus, $2 \cdot 3 = 2 \times 3 = 6$.

The division sign ($\div$) has the same meaning as in arithmetic. It is frequently replaced by the fraction line; thus $\frac{6}{3}$ means the same as $6 \div 3$ and in both cases the result or quotient is 2. The two dots above and below the line in the division sign ($\div$) indicate the position of the numerator and denominator in a fraction, or the dividend and divisor in division. Another sign which is sometimes used is the inclined fraction line (/); thus $6/3$ means the same as $6 \div 3$ or $\frac{6}{3}$. This form has the advantage of being compact and also allowing both dividend and divisor (or numerator and denominator) to be written or printed on the same line.

Parentheses ( ), brackets [ ], braces { }, and other enclosing signs are used to indicate that everything between the two signs is to be treated as a single quantity and any sign placed before it refers to everything inside as a whole and to every part of the complete expression inside. Thus, $(+a) \times (-b)$ means the product of the number *plus a* and the number *minus b*; (ab) means the single number which is the product of a and b, and $-(ab)$ means the negative of that single number. Similarly, $(+a) \div (-b)$ means *plus a* divided by *minus b* and (a/b) means the single number which is the quotient of a and b.

The expression $(a + b)$ means the single number which is the sum of the numbers a and b, and $(a - b)$ means the single number which is their difference. Similarly, $(a + b) + (a - b)$ means the sum of the number $(a + b)$ and the number $(a - b)$, and $(a + b) - (a - b)$ would mean the difference of the same two numbers. Thus, we may write $(7 - 3) \times (4 + 2) = 4 \times 6 = 24$, or $(a + b) \times (a - b) = (a + b)(a - b)$. Similarly, $(x + y) \div (a - b) = \dfrac{(x + y)}{(a - b)} = \dfrac{x + y}{a - b}$. This form does *not* mean that the sum $(x + y)$ or x and y each are divided by a and b separately, but that the *single number* which is the sum of x and y is divided by the *single number* which is the difference

between a and b. Thus, $\dfrac{12 + 3}{9 - 6} = (12 + 3) \div (9 - 6) = 15 \div 3 = 5.$

Another sign which is sometimes useful is the sign which means "greater than" or "less than." The sign ($>$) means "greater than" and the sign ($<$) means "less than." Thus, $a > b$ means that "a is greater than b," and $3 < 5$ means "3 is less than 5."

The sign $\therefore$, three dots at the corners of a triangle, means "hence" or "therefore."

As new subjects are taken up we shall occasionally need new signs. These will be explained when needed.

Chapter 2

ADDITION AND SUBTRACTION

8. Addition of Algebraic Numbers. If an arithmetical number is simply a number and refers to no particular thing it may be added to any other such number. Thus, 3, 7, 8 may be added to give 18. If, however, the numbers refer to particular things or kinds of things they cannot be added unless they all refer to the same thing. Thus, we can say that 3 airplanes, 7 airplanes and 8 airplanes are 18 airplanes; 3 boats, 7 boats and 8 boats are 18 boats; 3 cars, 7 cars and 8 cars are 18 cars. But we cannot say that 3 airplanes, 7 boats and 8 cars are 18 of anything.

It will be recalled that, in algebra, letters are used to represent numbers. Thus,

$3a + 7a$ means 3 times a number + 7 times the *same* number. For this reason, we can say that $3a + 7a = 10a$, or 3 times a number + 7 times the same number equals 10 times that number.

However, $3a + 7b$ means 3 times a certain number + 7 times a different number. In this case, we cannot combine $3a + 7b$. We say that $3a + 7b$ is an *indicated sum.*

In the manner just described, we can add $4x^2$ and $5x^2$ and the sum is $9x^2$, the x^2 ("x square") being the thing or number considered and the amount of it being in the one case four and in the other five. Similarly,

$$y^3 + 7y^3 + 2y^3 = 10y^3,$$

and

$$3z^4 + 9z^4 + 6z^4 = 18z^4,$$

but

$$4x^2 + y^3 + 3z^4$$

is an indicated sum and the addition cannot be carried further. This indicated sum is to be thought of as the single quantity $(4x^2 + y^3 + 3z^4)$. The different parts of an expression of this kind which are separated by plus (or minus) signs are called the *terms* of the complete expression.

Now the sum $2 + 3 = 5$, the number $1 + 6 + 4 = 11$, and

$7 + 9 + 12 = 28$ can all be added to give 44, and this addition can be written in either of the following terms:

$$5 + 11 + 28 = 44$$

$$(2 + 3) + (1 + 6 + 4) + (7 + 9 + 12) = 44$$

$$2 + 3 + 1 + 6 + 4 + 7 + 9 + 12 = 44.$$

That is, we can first perform the separate additions $2 + 3$, $1 + 6 + 4$, $7 + 9 + 12$ and then add the sums, or indicate the separate additions and then carry them out, or write out the continued indicated sum of all the numbers and find the final sum in a single addition.

In the same way,

$$(2a + 3a) + (a + 6a + 4a) + (7a + 9a + 12a)$$

is the same as

$$5a + 11a + 28a$$

and is equal to $44a$, or by removing the enclosing signs (parentheses),

$$2a + 3a + a + 6a + 4a + 7a + 9a + 12a = 44a,$$

and, similarly, for b, c, etc., provided all the separate numbers or *terms* are of the same kind or refer to the same thing.

Similarly we may add the three quantities $(2a + 3b)$, $(a + 6b + 4c)$ and $(7a + 9b + 12c)$, each of the indicated sums in parentheses being thought of as a single number. This may be written as

$$(2a + 3b) + (a + 6b + 4c) + (7a + 9b + 12c),$$

or, by removing the parentheses as above,

$$2a + 3b + a + 6b + 4c + 7a + 9b + 12c.$$

In this single complete indicated sum some of the terms are of the same kind and therefore some of the indicated additions can be carried out. Thus, the terms $2a$, a and $7a$ can be added; the terms $3b$, $6b$ and $9b$ can be added; and so can the terms $4c$ and $12c$. The *total algebraic sum* is, therefore, the sum of $10a$, $18b$ and $16c$, that is, the indicated sum $(10a + 18b + 16c)$. The entire addition may, therefore, be written out in the different forms:

$$(2a + 3b) + (a + 6b + 4c) + (7a + 9b + 12c),$$

$$2a + 3b + a + 6b + 4c + 7a + 9b + 12c,$$

$$(2a + a + 7a) + (3b + 6b + 9b) + (4c + 12c),$$

and in either case the total algebraic sum is the single number

$$10a + 18b + 16c.$$

The first form in which the separate quantities or indicated sums are grouped in parentheses suggests that as in arithmetic we may indicate and perform the entire addition as follows:

$$\begin{array}{r}
2a + 3b \phantom{{}+ 12c} \\
a + 6b + 4c \\
7a + 9b + 12c \\
\hline
10a + 18b + 16c
\end{array}$$

In this form, which is the most convenient when there are many terms, the separate terms containing a, b, c are written in separate columns as we write units, tens, hundreds, etc., in arithmetic, and each column is added. The total sum is then the indicated sum of the separate column sums.

One more example will serve to illustrate the method completely: Add $4a + 3x + 2y^2$, $3a + 5x + 7y^2$, $a + 2x + 4y^2$, and $2x + 3y^2 + 7z^3$. This is carried out as follows:

$$\begin{array}{rrrr}
4a + & 3x + & 2y^2 & \\
3a + & 5x + & 7y^2 & \\
a + & 2x + & 4y^2 & \\
 & 2x + & 3y^2 & + 7z^3 \\
\hline
8a + & 12x + & 16y^2 & + 7z^3
\end{array}$$

9. Addition of Negative Numbers. Let us now consider the addition of negative numbers and of positive and negative numbers. We have seen in article 5 that

$$-7 - 3 = -10$$

or,

$$-3 - 7 = -10$$

$$+3 - 3 = 0$$

$$+15 - 20 = -5$$

or,

$$-20 + 15 = -5.$$

Similarly, we could write 20 airplanes minus 15 airplanes equals 5 airplanes, or $+20a - 15a = 5a$; 15 dollars minus 20 dollars equals a deficit of 5 dollars, or $15d - 20d = -5d$. Also if a thermometer

reading is -3 degrees and there is a drop of 7 degrees the reading will be $-3 - 7 = -10$ degrees, the 7 degree drop being a change of -7 degrees. Here we are said to have *added* the -7 to the -3, as in ordinary language one might say that "to the 3 degrees below zero there was 'added on' a drop of 7 degrees" so that the thermometer stands at -10 degrees. Using the same form of writing and expression, the money example $15d - 20d = -5d$ may be written as

$$(+15d) + (-20d) = (-5d)$$

and the minus $20d$ $(-20d)$ is said to be *added* to the plus $15d$ $(+15d)$ to produce an *algebraic sum* of minus $5d$, $(-5d)$.

Similarly, we can write

$$(2a + 3b) + (7a - 2b + 8c) + (-4a + 5b - 6c)$$

as

$$2a + 3b + 7a - 2b + 8c - 4a + 5b - 6c$$

or,

$$2a + 7a - 4a + 3b - 2b + 5b + 8c - 6c$$

or,

$$[2a + 7a - 4a] + [3b - 2b + 5b] + [8c - 6c],$$

and this is the same as

$$[(2a + 7a) - 4a] + [(3b + 5b) - 2b] + [8c - 6c]$$

or,

$$[9a - 4a] + [8b - 2b] + [8c - 6c],$$

which gives as the *algebraic sum* of the positives *and* negatives,

$$5a + 6b + 2c.$$

What we have done here is to collect all *like terms* (terms containing the same symbol), combine the positives of each kind of term, and then combine the positive and the negative terms of that kind. This process is simply and easily carried out by writing *like terms in columns* as follows:

$$\begin{array}{r} 2a + 3b \phantom{{}+8c} \\ 7a - 2b + 8c \\ -4a + 5b - 6c \\ \hline 5a + 6b + 2c \end{array}$$

Similarly, the addition of the two algebraic numbers $(a - b - 16x)$ and $(-a - b + 11x)$ is carried out as follows:

$$\begin{array}{r} a - b - 16x \\ -a - b + 11x \\ \hline -2b - 5x \end{array}$$

One more example will illustrate the method completely: Add $2c + 3d^3 + 4w^2 + 3y$, $c - 8d^3 + 2w^2 - 11y$, $9d^3 + 6y + w^2 - 3c$, and $14y + 7d^3 - 11w^2$. This is carried out as follows, the terms being arranged in alphabetical order:

$$\begin{array}{rrrr} 2c + & 3d^3 + & 4w^2 + & 3y \\ c - & 8d^3 + & 2w^2 - & 11y \\ -3c + & 9d^3 + & w^2 + & 6y \\ & 7d^3 - & 11w^2 + & 14y \\ \hline 0 + & 11d^3 - & 4w^2 + & 12y, \end{array}$$

the total algebraic sum being

$$11d^3 - 4w^2 + 12y.$$

10. General Rules for Algebraic Addition. From the results of the preceding two articles it is seen at once that the general procedure in adding positive, negative, or positive and negative algebraic numbers, can be summarized in the following

RULE FOR ADDITION: (1) *Write all the terms on the same line without enclosing signs and each preceded by its specified sign; or arrange the terms in columns with each term preceded by its specified sign, like terms in the same column and unlike terms in different columns.*

(2) *Combine the coefficients of all positive and of all negative terms of each kind of term (or in each column) and then take the arithmetical difference of the two results.*

(3) *Write this difference as the coefficient of that same kind of term and place before it the sign of the larger of the two combined coefficients.*

(4) *The indicated sum of the resulting positive and negative sums is the total algebraic sum.*

In writing terms containing different letters of the alphabet, it is convenient to arrange them in alphabetical order. Thus, the expression $3b - 7x^2 - 4a + 5d + 2y$ would be written as $-4a + 3b + 5d - 7x^2 + 2y$.

In writing terms containing different powers of the same symbol it is convenient to write them in *descending order* of the exponents unless otherwise specified, that is, with the greatest exponent first, the next

smaller second, etc. Thus, $x^2 - x^4 + 2x^3 + 7x^6$ should be written $7x^6 - x^4 + 2x^3 + x^2$. The reverse order, $x^2 + 2x^3 - x^4 + 7x^6$ is called the *ascending order* and is only used in special cases.

Algebraic terms may consist of products as xy, $3ab$, $2x^2z$, etc. When such terms have the *same letters* with the *same exponents* they are like terms; otherwise, they are unlike terms. Thus, $3ab$ and $2ab$ are like terms, but $4a^2b$ and $7ab$ are unlike terms; x^2y and $2yx^2$ are like but $4x^3y$ and $6x^2y^2$ are unlike. The sign before such a term applies to the term *as a whole* and not to either individual letter.

If several product terms contain the same letter with different exponents, they should be arranged in the descending order of the exponents of that letter, any other letters in the terms then occurring in whatever order they may.

11. Algebraic Subtraction. The process of algebraic subtraction is based upon the same general principles as the process of arithmetic subtraction.

In performing the following subtraction

$$\begin{array}{l} 9 \text{ (minuend)} \\ \underline{5 \text{ (subtrahend)}} \\ 4 \text{ (difference)} \end{array}$$

we may say, "What must be added to 5 (the subtrahend) in order to obtain 9 (the minuend)?"

We will follow the same line of thinking in analyzing the next few examples. In order to keep a concrete picture in mind, we can regard the numbers as temperature readings. In each case, the operation to be performed is subtraction.

$$\begin{array}{r} +9 \\ \underline{+5} \\ +4 \end{array}$$

What must be added to $+5°$ to obtain $+9°$? It is clear that the thermometer must rise 4°. The difference is $+4$.

$$\begin{array}{r} +9 \\ \underline{-5} \\ +14 \end{array}$$

What must be added to $-5°$ to obtain $+9°$? Here, we start at 5° below zero and wish to reach a temperature of 9° above zero. Thus we must rise 5° to get to zero and then 9° more. The total rise is 14°. The difference is $+14$.

$$\begin{array}{r} -9 \\ \underline{+5} \\ -14 \end{array}$$

What must be added to $+5°$ to obtain $-9°$? Here, we start at 5° above zero and wish to reach a temperature of 9° below zero. Thus we must drop 5° to get to zero and then drop 9° more. The difference is -14.

$$\begin{array}{r} -9 \\ -5 \\ \hline -4 \end{array}$$

What must be added to $-5°$ to obtain $-9°$? Here, we must start 5° below zero and wish to reach a temperature of 9° below zero. Thus, we must drop 4°. The difference is -4.

Let us examine these results in more compact form.

$$\begin{array}{r} +9 \\ +5 \\ \hline +4 \end{array} \qquad \begin{array}{r} +9 \\ -5 \\ \hline +14 \end{array} \qquad \begin{array}{r} -9 \\ +5 \\ \hline -14 \end{array} \qquad \begin{array}{r} -9 \\ -5 \\ \hline -4 \end{array}$$

We notice that in each case, the correct result can be obtained by following this rule.

RULE FOR SUBTRACTION: *In subtracting one signed number from another, change the sign of the subtrahend and add.*

The process of subtraction is usually indicated in either of the following ways:

$$\text{Subtract: } \begin{array}{r} +7 \\ +3 \\ \hline \end{array} \quad \text{or } (+7) - (-3)$$

In the first case, the operation is indicated verbally. In the second case, the operation is indicated by the use of the symbol, $-$. Notice that the symbol $-$ is used in two ways, as an indication of subtraction, and as the sign of a signed number. When subtraction is indicated as $(+7) - (-3)$ the rule stated above is applicable.

As examples of algebraic subtraction:

$$(+14) - (+7) = +7$$

$$-9 - (-3) = -9 + 3 = -6$$

$$6a - (+4a) = 6a + 4a = 10a$$

$$-7x - (+3x) = -7x - 3x = -10x$$

In working with expressions containing more than one term, we follow the same general procedure. We change the sign of *every term* in the subtrahend, and proceed as in addition.

EXAMPLE: Subtract $(3b - c + 7x - y^2)$ from $(8b - 3x - c + 2y^2)$. This may be written as $(8b - 3x - c + 2y^2) - (3b - c + 7x - y^2)$.

SOLUTION:
$$\begin{array}{l} 8b - 3x - c + 2y^2 \\ 3b + 7x - c - y^2 \\ \hline 5b - 10x + 3y^2 \end{array}$$
or we may perform the subtraction without arranging the polynomials in columns.

$(8b - 3x - c + 2y^2) - (3b + 7x - c - y^2)$

$= 8b - 3x - c + 2y^2 - 3b - 7x + c + y^2$ (changing the signs of the terms in the subtrahend)

$= 5b - 10x + 3y^2$ (combining like terms).

12. Using and Removing Parentheses. In what follows the word "parentheses" will be used to represent either ordinary parentheses ( ), brackets [ ], braces { }, round brackets ⟦ ⟧, or any other enclosing signs.

From the last illustrative example in subtraction in the preceding article, we see that,

$$(8b - 3x - c + 2y^2) - (3b - c + 7x - y^2)$$

is the same as

$$8b - 3x - c + 2y^2 - 3b + c - 7x + y^2.$$

In this second form of the expression the parentheses are removed from both groups but the signs of the terms are changed in the second group of terms, the group preceded by the *minus* sign. Similarly, the complete expression

$$-(3a + 4b) + (7x - 2a) - (5b + 4y)$$

is the same as

$$-3a - 4b + 7x - 2a - 5b - 4y,$$

and here again in removing parentheses *preceded by a minus sign* the sign of each term inside the parentheses is changed but the signs of terms inside parentheses preceded by a plus sign remain unchanged. Also, since the two expressions

$$8b - 3x - c + 2y^2 - 3b + c - 7x + y^2$$

$$+(8b - 3x - c + 2y^2) - (3b - c + 7x - y^2)$$

are equivalent to each other, we see that when a group of terms is *enclosed* in parentheses preceded by a minus sign the sign of each term inside is changed since subtraction is to be performed. When the group is preceded by a plus sign the sign of each term inside is left unchanged.

Thus we have the

RULE FOR PARENTHESES: *A group of terms may be removed from or enclosed in parentheses by changing the sign of each term in the group when the*

parentheses are preceded by a minus sign. When the parentheses are preceded by a plus sign the signs inside are not changed.

This rule applies to both arithmetical and algebraic numbers.

It also applies to parentheses within parentheses. Thus, on removing all the parentheses from the expression

$$\{2b - \llbracket c - [7x - 8y + (2xy - yz^3) + 8d] - b + 5\rrbracket\}$$

we get, when the inner parentheses are removed,

$$\{2b - \llbracket c - [7x - 8y + 2xy - yz^3 + 8d] - b + 5\rrbracket\},$$

and on removing the brackets,

$$\{2b - \llbracket c - 7x + 8y - 2xy + yz^3 - 8d - b + 5\rrbracket\},$$

and on removing the round brackets,

$$\{2b - c + 7x - 8y + 2xy - yz^3 + 8d + b - 5\}.$$

When the braces (understood as preceded by plus) are removed, this becomes, finally,

$$2b - c + 7x - 8y + 2xy - yz^3 + 8d + b - 5.$$

The expression may now be simplified by combining any like terms according to the usual rules of algebraic addition.

In the example just given it is to be noted that by beginning with the inner parentheses and proceeding outward the signs of some of the terms were changed several times. This may be avoided by beginning with the outer parentheses and proceeding inward. Thus, on removing the braces, round brackets, brackets, and parentheses in turn the expression takes the successive forms:

$$\begin{aligned}
&\{2b - \llbracket c - [7x - 8y + (2xy - yz^3) + 8d] - b + 5\rrbracket\}\\
&2b - \llbracket c - [7x - 8y + (2xy - yz^3) + 8d] - b + 5\rrbracket\\
&2b - c + [7x - 8y + (2xy - yz^3) + 8d] + b - 5\\
&2b - c + 7x - 8y + (2xy - yz^3) + 8d + b - 5\\
&2b - c + 7x - 8y + 2xy - yz^3 + 8d + b - 5,
\end{aligned}$$

and the final form is the same as the previous result.

By proceeding inward in the manner just illustrated the entire set of parentheses may be removed at one operation and the final result written out at once by noticing the sign before each successive parenthesis and changing each sign inside that pair if the preceding sign is

minus and leaving it unchanged if the sign is plus. Examination of the first and last forms of the preceding example will show that this is the case.

A procedure similar to those illustrated above is followed in enclosing various groups of terms in parentheses of any kind.

13. Remarks. The fundamental algebraic methods developed in this chapter and the preceding chapter have been explained in great detail and in informal and non-technical language, and the reader should be sure that they are thoroughly understood before proceeding further. After these two chapters have been carefully read every one of the exercises in the following article should be worked out for practice and to fix the understanding of the methods. In mathematics as nowhere else, unless in music, nothing is more true than the adage that "practice makes perfect," and if one wishes to study science, engineering or higher mathematics the fundamental methods and principles of algebra must be mastered as perfectly as possible. The use of algebra must be as easy and natural as counting or the use of the multiplication table.

14. Exercises.

The following exercises are to worked out by the reader by applying the rules given in articles 10, 11 and 12.

Add the following:

1. $5a,\ 7a,\ 11a,\ a,\ 23a$.
2. $-3x,\ -5x,\ -7x,\ -x$.
3. $4b,\ 7b,\ -3b,\ -2b,\ b$.
4. $9x^2 + 11x^2 - 3x^2 - 4x^2 + x^2$.
5. $-5ab + 6bc - 7ac$; $8ab - 4bc + 3ac$; and $-2ab - 2bc + 4ac$.
6. $2a - 3b + c$; $15a - 21b - 8c$; $3a + 24b + 7c$.
7. $x^5 - 4x^4y - 5x^3y$; $3x^4y + 2x^3y^3$; $2x^5 - 6x^4y + x^3y^3$.
8. $3a^2 - \frac{2}{5}ab - \frac{1}{2}b^2$; $-\frac{3}{2}a^2 + 2ab - \frac{1}{2}b^2$; $\frac{3}{2}a^2 - \frac{3}{5}ab + b^2$.
9. $x^3 + 3x^2y + 3xy^2 + y^3$; $2y^3 - 3x^2y - 6xy^2 - x^3$; $3x^2y + 4xy^2$.
10. $a^3 - 4a^2b + 6abc$; $a^2b - 10abc + 2a^3$; $b^3 + 3a^2b + abc$.

Subtract the following:

11. $x^3 - x^2 + x + 1$, from $x^3 + x^2 - x + 1$.
12. $-4xy + 2yz - 10zx$, from $3xy - 5yz + 8zx$.
13. $-10a^2b^2 + 15ab^2 - 8a^2b$, from $10a^2b^2 + 15ab^2 + 8a^2b$.
14. $7xy^2 - y^3 - 3x^2y + 5x^3$, from $8x^3 + 7x^2y - 3xy^2 - y^3$.
15. $\frac{1}{2}x^2 - \frac{1}{3}xy - \frac{3}{2}y^2$, from $\frac{3}{2}x^2 + xy - y^2$.
16. $\frac{2}{3}a^2 - \frac{5}{2}a - 1$, from $-\frac{2}{3}a^2 + a - \frac{1}{2}$.
17. $1 - x + x^5 - x^4 - x^3$, from $x^4 - 1 + x - x^2 + x^5$.
18. $5a^2b - 7ab^2 + 6cd$, from $-7a^2b + 8ab^2 + cd$.

Remove the parentheses in the following and if possible simplify:

19. $a - (b - c) + a + (b - c) + b - (c + a)$.
20. $a - [b + \{a - (b + a) + c\} - c]$.
21. $-5x - [3y - \{2x - (2y + x) + y\}]$.
22. $3a - [a + b - \{a + b + c - (a + b + c + d)\}]$.
23. $-[a + \{a - (a - x) - (a + x) - a\} - a]$.
24. $-[a - \{a + (x - a) - (x - a) - a\} - 2a]$.
25. $2x^2 - \{x^2 + 3ab + [xy^3 - 2ab] + 2xy^3\} + ab + 2$.

Chapter 3

MULTIPLICATION AND DIVISION

15. Multiplication of Monomials. As previously defined the *terms* of an algebraic expression are the parts which are separated by plus and minus signs. Thus in the expression $a + 2b - \frac{1}{2}c^2 - x^2y^3$ the terms are a, $2b$, $-\frac{1}{2}c^2$ and $-x^2y^3$, the sign in each case belonging with and forming a part of the term.

If an algebraic expression consists of one term it is called a *monomial*. An expression consisting of two terms is a *binomial*, and one of three terms is a *trinomial*. (These names come from the Greek and Latin words for "one," "two" and "three.") Any expression containing more than one term (two, three, four, etc.) is called a *polynomial* (from the Greek word for "several" or "many.")

In this article we discuss the methods and rules of algebraic multiplication of monomials.

We have already seen that an algebraic symbol may be multiplied by an ordinary number, as two times x equals $2x$. Similarly, any two symbols may be multiplied together and the multiplication sign is usually omitted, as $a \times b = ab$. As seen in article 6, if one factor is positive and the other is negative the product is negative but if both factors are positive or both are negative the product is positive. Thus, $(+x) \times (-y) = -(xy)$, $(+a) \times (+b^2) = +(ab^2)$ or simply ab^2, $(-2) \times (-y^3) = +(2y^3)$ or simply $2y^3$.

Obviously, since 2 times 3 dollars is 6 dollars, we may write in symbols $2 \times 3d = 6d$. That is, if a symbol has a numerical coefficient and the monomial consisting of this symbol and its coefficient (in the present case, $3d$) is to be multiplied by a numerical multiplier, the *coefficient alone* is multiplied. Thus, $4 \times 3x = 12x$, $5 \times (-7y^3) = -35y^3$, $(-4) \times (-8x^2) = 32x^2$, etc.

Also, since $x \cdot y = xy$ then $x \cdot 2y = 2xy$. That is, when multiplying by symbols or literal numbers the coefficients remain unchanged and only the letters are multiplied. Thus, $a \times (-3b^4) = -3ab^4$, $(-x) \cdot (-7y^3) = 7xy^3$.

In multiplying several factors it is immaterial which is taken first,

as $2 \times 3 \times 5 = 3 \times 2 \times 5 = 5 \times 3 \times 2$, etc. Similarly, $2 \cdot a \cdot x = (2a) \cdot x = 2 \cdot (ax) = a \cdot (2x)$, etc., and the product is $2ax$. In the same way, it is seen at once that $2a \times 3b = (2a \times 3) \times b = (6a) \times b = 6ab$. That is, in multiplying monomials when *both* have numerical coefficients, multiply the numerical coefficients to form the coefficient of the product, and multiply the literal numbers to form the literal part of the product. Thus, $(4x) \cdot (7y^2) = 28xy^2$, $(-3ab) \times 4d = -12abd$, $(-7c^2) \times (-8z^3) = 56c^2z^3$.

Suppose we have to multiply $2a$ by $4a$. According to the above results, this gives $8aa$, but as seen in articles 2 and 3, $aa = a^2$. Therefore, $2a \times 4a = 8a^2$. Suppose that we have to multiply $8a^2$ by $4a^3$; the product is $32a^2a^3$. But $a^2 = aa$ and $a^3 = aaa$; hence, $a^2a^3 = (aa) \times (aaa) = aaaaa$ which is a^5. That is, $a^2a^3 = a^5$. In the same way $x^3 \cdot x^4 = x^7$, $4y^4 \cdot 8y^5 = 32y^9$, etc. From these results it is now plain that in order to multiply together two different powers of the *same letter* we simply write the same letter with a new exponent which is the *sum of the two original exponents.* Thus $\frac{1}{2}x^2 \cdot (-6x^8) = -3x^{10}$. Similarly, $a^2 \times a^5 \times a^3 = a^2 \times (a^5 \times a^3) = a^2 \times a^8 = a^{10}$. But $a^{10} = a^{2+5+3}$; therefore, the product of any number of factors containing the *same* letter or symbol raised to different powers, is found by simply *adding the exponents* to form the exponent of the same letter in the product. When no exponent is written it is understood to be 1, as $a^1 = a$.

As examples of such multiplication, therefore: $2a^2b \times 3a^3 = 6a^5b$; $5ab \times 4a^2b^2 = 20a^3b^3$; $(-7xy^2) \cdot x^2y^2 = -7x^3y^4$; $(-4a^4z^7) \times (-3a^5z^2) = 12a^9z^9$; etc.

The results and illustrations of this article can be summed up in the

RULE: *The product of monomials is found by multiplying the numerical coefficients to form the coefficient of the product, multiplying the literal symbols to form the literal part, and using the algebraic rule of signs. If the same letter occurs in two or more factors add its several exponents to form the exponent of the same letter in the product.*

16. Multiplication of Polynomials. Consider the binomial expression $5 - 2$, and let it be multiplied by 4. This is written $4 \times (5 - 2)$, or more simply $4(5 - 2)$. Now, $5 - 2 = 3$, and hence $4(5 - 2) = 4 \times 3 = 12$. Suppose that instead of first subtracting the 2 from the 5 and then multiplying, we proceed as follows: $4(5 - 2) = (4 \times 5) - (4 \times 2)$, that is, first multiply each term of the binomial separately by the multiplier and then subtract. The

result is seen to be $20 - 8 = 12$, which is the same as that obtained at first. Similarly, $3(8 - 4 + 5) = 3 \times 9 = 27$, or $3(8 - 4 + 5) = 24 - 12 + 15 = 27$. We see that either method gives the correct result.

Let us now consider a binomial in which one term is literal and one numerical, as $5 + a$, and multiply it by 4. In this case, we cannot use the first method used above because $5 + a$ is only an indicated sum and the addition cannot actually be performed. We can, however, use the second method. This method gives as the product $4(5 + a) = 20 + 4a$. Similarly, $2(a + b) = 2a + 2b$; $-4a(b + 3y) = [(-4a) \times b] + [(-4a) \times (3y)] = -4ab - 12ay$. In the same way, we find $xy(3b + y^2 - 4a) = 3bxy + xy^3 - 4axy$. These examples will serve to show that to multiply a polynomial by a monomial, multiply *each term* of the polynomial by the monomial according to the rule given in the preceding article, observing the rule of signs.

Next consider the product $(3 + 2)(8 - 4)$. This is, of course, $5 \times 4 = 20$. But according to the method used above this may also be carried out by multiplying the 8 and the -4 of the multiplicand, separately, by the multiplier $(3 + 2)$. This operation gives $[(3 + 2) \times 8] - [(3 + 2) \times 4]$. But considering each of these two resulting products, separately, each is seen to be a binomial multiplied by a monomial. The first is, $(3 + 2)8 = (3 \times 8) + (2 \times 8)$ and the second is $-[(3 \times 4) + (2 \times 4)]$ or, removing the brackets, $-(3 \times 4) - (2 \times 4)$. The complete product is, therefore,

$$(3 + 2)(8 - 4) = (3 \times 8) + (2 \times 8) - (3 \times 4) - (2 \times 4),$$

and the result is $24 + 16 - 12 - 8 = 20$, the same as before. Examining the complete product it is seen to consist of four terms, formed by *multiplying each term of the multiplicand separately by each term of the multiplier* and combining the results according to their signs.

Applying this method to expressions whose terms contain literal numbers we have, for example, $(a + b)(x + y) = ax + ay + bx + by$; $(2a + c)(x - 3y + 5d) = 2ax - 6ay + 10ad + cx - 3cy + 5cd$; $(a - 2b + 3c)(a^2 + xy - cy^2) = a^3 + axy - acy^2 - 2a^2b - 2bxy + 2bcy^2 + 3a^2c + 3cxy - 3c^2y^2$; etc.

Summarizing these results and methods we have the following

RULE: *The product of any two polynomials is found by multiplying each term of one by each term of the other and adding the results, algebraically.*

In applying this rule it is usually more convenient to take as the

multiplier the polynomial containing the smaller number of terms.

If either of the factors contains more than two terms it is usually more convenient to write the multiplier underneath the multiplicand, draw a line under the multiplier and proceed as in arithmetic, multiplying each term of the multiplicand, separately, by each term of the multiplier in turn, according to the above rule, and adding the results in columns of like terms, if any.

Two examples are given below:

Find the product $(2x^2 + y^3)(3x^3 - 4y^2 + z)$.

$$\begin{array}{r}
3x^3 - 4y^2 + z \\
2x^2 + y^3 \\
\hline
3x^3y^3 - 4y^5 + y^3z \\
6x^5 - 8x^2y^2 + 2x^2z \qquad\qquad\qquad\qquad\qquad \\
\hline
6x^5 - 8x^2y^2 + 2x^2z + 3x^3y^3 - 4y^5 + y^3z
\end{array}$$

Arranging this product in alphabetical order and in descending order of the exponents of x, it is:

$$6x^5 + 3x^3y^3 - 8x^2y^2 + 2x^2z - 4y^5 + y^3z.$$

Multiply $4a^4 + 3a^2b - 2b^2$ by $2a^2 - 3b$.

$$\begin{array}{rrrr}
 & 4a^4 & +\ 3a^2b & -\ 2b^2 \\
 & & 2a^2 & -\ 3b \\
\hline
 & -\ 12a^4b & -\ 9a^2b^2 & +\ 6b^3 \\
8a^6 & +\ 6a^4b & -\ 4a^2b^2 & \\
\hline
8a^6 & -\ 6a^4b & -\ 13a^2b^2 & +\ 6b^3
\end{array}$$

17. Some Special Products of Binomials. Let us find the product $(x + y)(x + y)$. Carrying out the multiplication as above:

$$\begin{array}{rrr}
 & x & +\ y \\
 & x & +\ y \\
\hline
 & xy & +\ y^2 \\
x^2 & +\ xy & \\
\hline
x^2 & +\ 2xy & +\ y^2
\end{array}$$

Similarly, $(2a + 3b)(2a + 3b) = 4a^2 + 12ab + 9b^2$.

But $(x + y)(x + y)$ is $(x + y)$ multiplied by itself, that is, the square of $x + y$, or $(x + y)^2$. Therefore, $(x + y)^2 = x^2 + 2xy + y^2$, or,

$$(x + y)^2 = x^2 + y^2 + 2xy. \tag{1}$$

Also, $(2a + 3b)(2a + 3b)$ is $(2a + 3b)^2$ and in the product, as given above, it is to be noted that $4a^2 = 2a \times 2a = (2a)^2$, $9b^2 = 3b \times 3b = (3b)^2$, and $12ab = 2(6ab) = 2(2a \times 3b)$. Therefore, we can write

$$(2a + 3b)^2 = (2a)^2 + (3b)^2 + 2(2a)(3b). \tag{2}$$

Comparing the results numbered (1) and (2) it is seen that they are of the same general form, that is, the *square of the sum* of two terms. The result in each case is seen to consist of the *square of the first term*, plus the *square of the second term*, plus *twice the product of the two terms.*

It is especially to be noted that $(x + y)^2$ is NOT $x^2 + y^2$, but $(x^2 + y^2) + 2xy$.

If we multiply $x - y$ by $x - y$ in the same way we find $(x - y)(x - y) = x^2 - 2xy + y^2$, or

$$(x - y)^2 = x^2 + y^2 - 2xy. \tag{3}$$

Comparing the results (1) and (3) we can state in words that: *The square of the* SUM *of two terms is the sum of their separate squares* PLUS *twice their product, and the square of the* DIFFERENCE *of two terms is the sum of their squares* MINUS *twice their product.*

Applying the last rule to find the square of $3a^2 - xy$ we have at once without carrying out the multiplication:

$$\begin{aligned}(3a^2 - xy)^2 &= (3a^2)^2 + (xy)^2 - 2(3a^2)(xy)\\ &= 9a^4 + x^2y^2 - 6a^2xy.\end{aligned} \tag{2a}$$

Similarly, the square of the sum or difference of any two monomials can be found without carrying out the multiplication, the results (2) and (2a) being illustrations of both forms.

Consider next the product $(x + y)(x - y)$. Multiplying term by term, according to the rule for multiplication of polynomials, we get $x^2 - xy + xy - y^2$ and the $-xy$ and $+xy$ terms cancel. The product is, therefore,

$$(x + y)(x - y) = x^2 - y^2. \tag{4}$$

This result is different from either (1) or (3) above. In (1) we have $(x + y)(x + y)$, the product of the sum by the sum, and in (3) we have $(x - y)(x - y)$, the product of the difference by the difference, of the same two terms. In (4), however, we have $(x + y)(x - y)$, the product of the sum by the difference of the same two terms. Result

(4) is stated in words as follows: *The product of the sum and difference of two terms equals the* DIFFERENCE *of their squares.*

As an example of the application of this rule, let us write out without multiplication the product of $3a^2b + 7xy^3$ by $3a^2b - 7xy^3$. We have

$$\begin{aligned}(3a^2b + 7xy^3)(3a^2b - 7xy^3) &= (3a^2b)^2 - (7xy^3)^2 \\ &= 9a^4b^2 - 49x^2y^6.\end{aligned}$$

The results (1), (3) and (4), or the corresponding statements in words are of great importance and are very frequently used in algebra and its applications, and should be thoroughly understood and remembered.

As another special product of binomials, let us consider the product $(a + b)(a + b)(a + b)$. Here the binomial $a + b$ is used three times as a factor; the product is, therefore, the third power or cube of $a + b$, or $(a + b)^3$. In order to find its value we can write it as $(a + b)[(a + b)(a + b)]$, and the value of the product in square brackets is of the form (1) above. Therefore, $(a + b)(a + b) = a^2 + 2ab + b^2$ and the full product can be written $(a + b)^3 = (a + b)[(a + b)^2] = (a + b)(a^2 + 2ab + b^2)$. Carrying out the last indicated multiplication we have:

$$\begin{array}{rrrr} & a^2 & + 2ab & + b^2 \\ & & a & + b \\ \hline & a^2b & + 2ab^2 & + b^3 \\ a^3 & + 2a^2b & + ab^2 & \\ \hline a^3 & + 3a^2b & + 3ab^2 & + b^3 \end{array}$$

That is,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. \tag{5}$$

In order to find the product $(a + b)(a + b)(a + b)(a + b)$, or $(a + b)^4$, we only have to remember that according to the rule for exponents found in article 15, $(a + b)^4 = (a + b) \times (a + b)^3$, and we already have $(a + b)^3$ written out in (5) above. Therefore, we only have to multiply

$$(a + b)(a^3 + 3a^2b + 3ab^2 + b^3)$$

and when this is done it is found that

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \tag{6}$$

In the same way, we could find $(a + b)^5$, $(a + b)^6$ or *any power* of the *binomial* $(a + b)$ by continued multiplication. The process is very tedious, however, and we shall later develop a rule for writing out at once any power of any binomial.

Following the method just used to find $(a + b)^3$ we can find $(a - b)^3$ by using the result (3). It turns out to be

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3. \tag{7}$$

Similarly,

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4. \tag{8}$$

The general rule for powers of binomials which we shall find later can also be applied to $(a - b)^3$, $(a - b)^4$, $(a - b)^5$, etc., and (7) and (8) above will not be worked out in detail here. The products (6), (7) and (8) should be worked out by the reader, however, and will provide excellent practice in multiplication.

A very useful application of the forms (1) and (3) is the finding of squares of numbers too large to be handled by the ordinary multiplication table. The table gives us, of course, $9^2 = 9 \times 9 = 81$, $10^2 = 10 \times 10 = 100$, $11^2 = 11 \times 11 = 121$, $12^2 = 12 \times 12 = 144$. Similarly, we can find by extending the table that $13^2 = 13 \times 13 = 169$, $14^2 = 196$, $15^2 = 225$, etc., but when we come to numbers like 19, 21, 38, 79, 83, 98, etc., we usually have to write them down and multiply out. By the use of the rules (1) and (3), however, this can be avoided. First let us consider $12^2 = 144$. We can write 12 as $10 + 2$ and then $12^2 = (10 + 2)^2$. Applying our rule (1) to this binomial we have

$$(10 + 2)^2 = 10^2 + 2^2 + 2(10 \times 2) = 100 + 4 + 40 = 144.$$

Similarly, $19 = 20 - 1$ and $19^2 = (20 - 1)^2$. Applying (3) this gives

$$(20 - 1)^2 = 20^2 + 1^2 - 2(20 \times 1) = 400 + 1 - 40 = 361.$$

In the same way:

$$\begin{aligned} 83^2 &= (80 + 3)^2 \\ &= 80^2 + 3^2 + 2(80 \times 3) = 6400 + 9 + 480 = 6889 \\ 98^2 &= (100 - 2)^2 \\ &= 100^2 + 2^2 - 2(100 \times 2) = 10000 + 4 - 400 = 9604. \end{aligned}$$

With a little practice such multiplication can be performed mentally and very rapidly without putting pen to paper, and the square of any number less than 100 found at once.

18. Division of Monomials. Since $2 \times 3ab = 6ab$ we know at once that $6ab \div 2 = 3ab$. Also, $6ab \div 3 = 2ab$. Similarly, $-12y^2 \div 4 = -3y^2$ and $12y^2 \div (-3) = -4y^2$, according to the rule of signs for quotients (article 6). From these illustrations we have at once: To divide a monomial consisting of letters with a numerical coefficient, by a number, divide the *coefficient* of the dividend by the divisor, and apply the rule of signs. As examples: $16ab^2 \div 4 = 4ab^2$; $(-12z^3) \div (-3) = 4z^3$. These can be written $\frac{16ab^2}{4} = 4ab^2$; $\frac{-12z^3}{-3} = 4z^3$, and we simply cancel common factors above and below the fraction line and apply the rule of signs. Thus, $24x^2y \div (-18) = \frac{24x^2y}{-18} = -\frac{4x^2y}{3} = -\frac{4}{3}x^2y$. When some of the common factors are literal factors (letters) the same rule holds good. For example: $32ab \div 4a = 8b$, or $\frac{32ab}{4a} = 8b$ (which is obvious because $4a \times 8b = 32ab$); $\frac{-56xyz}{-7xz} = 8y$; $\frac{14a^2bc}{-21a^2b} = -\frac{2c}{3} = -\frac{2}{3}c$.

Suppose now that some of the factors of dividend and divisor consist of the *same letter* with the same or different exponents. Consider for example $\frac{a^5bc}{a^2}$. We know that $a^5 = a^2 \times a^3$, or a^2a^3. Therefore, $\frac{a^5bc}{a^2} = \frac{a^2a^3bc}{a^2} = a^3bc$. Similarly, $\frac{14c^7f}{-2c^5f} = \frac{14c^5c^2f}{-2c^5f} = -7c^2$. Examining these illustrations we see that in the first $a^3 = \frac{a^5}{a^2} = a^{(5-2)}$ and in the second $c^2 = \frac{c^7}{c^5} = c^{(7-5)}$. Similarly, $\frac{x^{11}}{x^4} = x^{(11-4)} = x^7$, etc. In the same way, $64a^2b^2x \div 8a^2b^5x$, or, $\frac{64a^2b^2x}{8a^2b^5x} = \frac{8a^2b^2x}{a^2b^2b^3x} = \frac{8}{b^3}$; $\frac{y^3}{y^7} = \frac{1}{y^4}$; etc.

From these illustrations we see at once that when some factors in dividend and divisor consist of the same *letters* with the same or different exponents, we cancel the one with the smaller exponent and subtract this exponent from the larger exponent of the same letter, and

obtain as a factor in the dividend or divisor of the final quotient the same letter with a new exponent.

Summarizing these results we can state the following

RULE: *To find the quotient of two monomials divide coefficients to find the coefficient of the quotient, and cancel common literal factors in dividend and divisor. If the same letter appears as a factor of both dividend and divisor with different exponents take the difference of these exponents as the exponent of the same letter in the quotient.*

As final illustrations of the application of this rule:

$$\frac{18x^2y^6z}{6x^6y^2z} = \frac{3y^4}{x^4};\ (-4a^3b^2c) \div (12a^3b^5c^2) = \frac{-4a^3b^2c}{12a^3b^5c^2} = -\frac{1}{3b^3c}.$$

19. Division of Polynomials. In article 16 we found that multiplication of algebraic polynomials can be carried out in exactly the same way as the multiplication of large numbers in arithmetic. The same is true of the division of polynomials. If a polynomial is to be divided by a monomial we use what might be called "short division" and separately divide *each term* of the dividend in turn by the monomial divisor.

Thus, in order to divide 91266 by 3 we have by arithmetical short division: $\dfrac{3\overline{)91266}}{30422}$. Similarly, to divide the polynomial $4a^3b^2c + 12ab^2c^3 - 18a^4b^4c^4$ by the monomial $2abc$ we have by algebraic "short division":

$$\frac{2abc\overline{)4a^3b^2c + 12ab^2c^3 - 18a^4b^4c^4}}{2a^2b + \quad 6bc^2 - \quad 9a^3b^3c^3}.$$

This division may also be carried out as follows, and this form of writing it is sometimes useful:

$$\frac{4a^3b^2c + 12ab^2c^3 - 18a^4b^4c^4}{2abc} = \frac{4a^3b^2c}{2abc} + \frac{12ab^2c^3}{2abc} - \frac{18a^4b^4c^4}{2abc}$$
$$= 2a^2b + 6bc^2 - 9a^3b^3c^3.$$

In dividing a polynomial by a polynomial, however, it is better to use long division. In order to understand algebraic long division better, let us first carry out a few ordinary long divisions in arithmetic. We first divide 6573 by 21, the division, of course, being exact and the quotient 313.

$$
\begin{array}{r}
21\overline{)6573}(313 \\
\underline{63} \\
27 \\
\underline{21} \\
63 \\
\underline{63}
\end{array}
$$

In order to see more clearly the meaning of the procedure in algebraic long division we will carry out this same division in a slightly different manner. We can write 6573 in the form $6000 + 500 + 70 + 3$ and 21 as $20 + 1$. The division then runs as follows:

$$
\begin{array}{l}
20 + 1)6000 + 500 + 70 + 3(300 + 10 + 3 \\
\qquad\quad \underline{6000 + 300} \\
\qquad\qquad\qquad 200 + 70 \\
\qquad\qquad\qquad \underline{200 + 10} \\
\qquad\qquad\qquad\qquad\quad 60 + 3 \\
\qquad\qquad\qquad\qquad\quad \underline{60 + 3}
\end{array}
$$

The quotient is $300 + 10 + 3 = 313$, as before.

In this form, the dividend 6573 is the polynomial $6000 + 500 + 70 + 3$, the divisor 21 is the binomial $20 + 1$, and the quotient 313 is the trinomial $300 + 10 + 3$. In writing the work out in this form it is to be noted that we divide the *first* term of the dividend by the *first term* of the divisor to get the first term of the quotient. *Both terms* of the divisor are then multiplied by this first term of the quotient and the resulting terms written in the same columns with the *like terms* of the dividend. These terms are then subtracted and the next term of the dividend is written down to give the two terms $200 + 70$. These two terms are now used as a dividend and the same steps are repeated. This process is continued until there are no more terms in the dividend to copy down. If there is then no remainder the division is exact. If there is a remainder it is written over the divisor to form a fraction and this fraction is added to the quotient already obtained to form the complete quotient. This procedure is followed *exactly* in algebraic long division.

As an illustration of algebraic division, let us divide the polynomial $6a^4b^2 + 7a^3b^3 + 4a^2b^4 + ab^5$ by the binomial $2ab^2 + b^3$.

$$\begin{array}{l}
2ab^2 + b^3)6a^4b^2 + 7a^3b^3 + 4a^2b^4 + ab^5(3a^3 + 2a^2b + ab^2 \\
\quad\quad\quad\quad \underline{6a^4b^2 + 3a^3b^3} \\
\quad\quad\quad\quad\quad\quad 4a^3b^3 + 4a^2b^4 \\
\quad\quad\quad\quad\quad\quad \underline{4a^3b^3 + 2a^2b^4} \\
\quad\quad\quad\quad\quad\quad\quad\quad 2a^2b^4 + ab^5 \\
\quad\quad\quad\quad\quad\quad\quad\quad \underline{2a^2b^4 + ab^5}
\end{array}$$

Let us next divide 6775 by 25. Writing dividend and divisor as polynomials we have:

$$\begin{array}{l}
20 + 5)6000 + 700 + 70 + 5(300 - 40 + 10 + 1 \\
\quad\quad\quad \underline{6000 + 1500} \\
\quad\quad\quad\quad\quad -800 + 70 \\
\quad\quad\quad\quad\quad \underline{-800 - 200} \\
\quad\quad\quad\quad\quad\quad\quad 200 + 70 \\
\quad\quad\quad\quad\quad\quad\quad \underline{200 + 50} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad 20 + 5 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{20 + 5}
\end{array}$$

In this case, in changing signs to subtract 1500 from 700, we get -800, and $-800 \div 20 = -40$. Then, $-40(20 + 5) = -800 - 200$. In subtracting the -200 from 70, the -200 becomes $+200$, but, since 200 and 70 cannot go in the same columns (one being hundreds and the other tens), they form the two terms of the new dividend $200 + 70$ and it is not necessary to write down the next term, 5, of the dividend until the next step. We then have $70 - 50 = 20$ and writing down the $+5$ we have $20 + 5$, and the first term, 20, of this new partial dividend divided by the first term, 20, of the divisor gives the last term, 1, of the quotient. The quotient is the polynomial $300 - 40 + 10 + 1 = 271$, which is correct, as may be verified.

The interest in this example lies in the features just described and in the fact that while all the terms of both dividend and divisor are positive one term in the quotient is negative. Division of arithmetical polynomials like the two already discussed may be carried out in exactly the same manner when any of the terms in either or both dividend and divisor are negative. This is true also of algebraic polynomials and is illustrated in the following algebraic example.

$$\begin{array}{r}
a^2 - 2ab + b^2)\overline{a^3 - 3a^2b + 3ab^2 - b^3}(a - b \\
\underline{a^3 - 2a^2b + ab^2} \\
- a^2b + 2ab^2 - b^3 \\
\underline{- a^2b + 2ab^2 - b^3}
\end{array}$$

Summarizing the procedure and results of the last four examples we have for division of polynomials by polynomials the following

RULE: (1) *Write both dividend and divisor in alphabetical order of the letters and descending order of the exponents of one of the letters.*

(2) *Divide the first term of the dividend by the first term of the divisor and write the quotient as the first term of the required quotient.*

(3) *Multiply the complete divisor by the quotient just obtained and write the resulting terms under the like terms of the dividend.*

(4) *Subtract the terms just written down from those above them and with the remainder write down the next term or terms of the dividend to make as many terms as there are in the divisor.*

(5) *Using the new terms as a dividend repeat the steps* (2), (3) *and* (4).

(6) *Continue this process until there are no more terms in the dividend to write down. If the last remainder is zero the division is complete and exact.*

(7) *If there is a remainder when the division is completed write the divisor beneath it to form a fraction and write this fraction as the last term of the quotient.*

(8) *All subtraction in the process is algebraic and in all division and multication the rules of signs apply.*

20. Multiplication and Division of Fractions. When the algebraic multiplication and division of monomials and polynomials is mastered, the algebraic multiplication and division of fractions will be found simple, the rules being exactly the same as in arithmetic, and the handling of the algebraic numbers the same as in the several preceding articles.

Fractions in algebra may have for numerator or denominator, or both, arithmetical numbers or algebraic monomials, polynomials or products. For example, $\frac{a}{b}$, $\frac{2x^2y}{z}$, $\frac{-(3ab + c^2)}{-2x}$, $\frac{ab^2 + 3c^3 - 4yz}{2a + 3x}$, $\frac{(a + b)^2}{(x + y)(x + z)}$ are algebraic fractions.

To multiply a fraction by a number, monomial or polynomial, multiply the *numerator* of the fraction by the multiplier and write the

product as the numerator of a new fraction which has the denominator of the original fraction for its denominator. This new fraction is the required product. If any factor appears in both numerator and denominator of this fraction it is to be cancelled.

To divide a fraction by a number, monomial or polynomial, multiply the *denominator* of the fraction by the divisor and write the product as the denominator of a new fraction which has for numerator the numerator of the original fraction. This new fraction is the required quotient and may be simplified by cancelling any common factors in numerator and denominator.

To multiply a fraction by a fraction multiply the numerators together to form the numerator of the product, and multiply the denominators together to form the denominator of the product. Any common factors in numerator and denominator of the product may be cancelled.

To divide a fraction by a fraction invert the divisor (interchange numerator and denominator) and multiply the dividend by the new fraction thus formed.

In all the multiplications of this article the methods already explained for multiplying monomials and polynomials are to be used and the rules of signs of multiplication and division are to be followed.

21. Exercises.

Multiply the following:

1. $2ab^2 \times 6$; $3a \times 2b$; $3ab \times 4c$; $-4a \times 3b$.
2. $7xy \times (-2)$; $3x \cdot 4x$; $\frac{1}{2}ax \cdot 5a$; $(-4xy) \times 3y^2$.
3. $2xy - 3yz + 7zx$ by 4.
4. $x - y - z$ by -3.
5. $3ab^2 - 3a^2b$ by -3.
6. $-ab + bc - ca$ by $-abc$.
7. $(5x^2y - 6xy^2 + 8x^2y^2) \times (3xy)$.
8. $x + 5$ by $x + 10$.
9. $(x + 5)(x - 5)$.
10. $(2x - 3)(x + 8)$.
11. $(ax + by)(ax - by)$.
12. $(2pq - 3r)(pq + 2r)$.
13. $3x^2 - 2x - 6$ by $2x - 5$.
14. $(a - b + 3c) \times (a + 2b)$.
15. $(a + b + c) \times (a + b - c)$.
16. $(x^2 + xy + y^2)(x - y)$.
17. $x^3 + 7x - 5$ by $x^2 - 2x + 3$.
18. $(a^3 + 2a^2b + 2ab^2) \times (a^2 - 2ab + 2b^2)$.

Divide the following:

19. $3ab$ by b; $12abc$ by $3ac$; $16xy$ by $-4x$.
20. $-18yz$ by $3z$; $-28xyz$ by $-7xy$; $-xyz$ by $-y$.
21. $3x^3$ by x; $27x^4$ by $-9x^3$; $-35x^6$ by $7x^3$.
22. $50y^3x^3 \div 5x^3y$; $(-48a^4) \div (-8a^3)$; $(77a^2bc) \div (-7a^2b)$.
23. $\dfrac{16b^2x^2y}{-2xy}$; $\dfrac{-63a^7b^8c^3}{-9a^5b^5c^3}$; $12a^6b^6c^6/3a^4b^2c$.
24. $x^3 - 3x^2 + x$ by x.
25. $y^6 - 7y^5 + 4y^4$ by y^2.
26. $(a^3 - a^2b - a^2b^2) \div a^2$.
27. $\dfrac{a^2 - ab + ac}{a}$.
28. $(34x^3y^2 - 51x^2y^3)/17xy$.
29. $\dfrac{4x^4y^4 - 8x^3y^2 + 6x^2y^3}{-2x^2y^2}$.
30. $x^2 + 11x + 30$ by $x + 6$.
31. $24x^2 - 65xy + 21y^2$ by $8x - 3y$.
32. $(x^2 + 3x + 2) \div (x + 1)$.
33. $(x^2 - 7x + 12) \div (x - 3)$.
34. $(a^2 - 11a + 30)/(a - 5)$.
35. $\dfrac{3x^2 - x - 14}{x + 2}$.
36. $x^3 - x^2 - 9x - 12$ by $x^2 + 3x + 3$.

Perform the following indicated operations:

37. $\dfrac{2a}{3} \times \dfrac{3}{b}$; $\dfrac{24x^2y}{a^2b} \times \dfrac{3a}{72x^2}$; $\dfrac{3(a+b)}{14x} \times \dfrac{7x^2}{9(a+b)^2}$.
38. $\dfrac{18xy}{5} \div 6x$; $\dfrac{25abc}{3xyz} \div \dfrac{5ab}{9xy}$; $\dfrac{x^2 + 2xy + y^2}{4} \div \dfrac{x + y}{4}$.
39. $\dfrac{x^3 - y^3}{14} \div \dfrac{x^2 + xy + y^2}{7} \times \dfrac{4}{x - y}$.
40. $\dfrac{(a - b)b}{a^2 - 2ab + b^2} \times \dfrac{xy(a^2 - b^2)}{a^2 + 2ab + b^2} \div \dfrac{b^2xy}{a^2 + ab}$.

Chapter 4

FACTORIZATION

22. Factors and Factoring. The word "factor" has the same meaning in algebra as in arithmetic. In general, if an algebraic expression of any form is the product of several other expressions, then each of those several expressions is a *factor* of the complete expression. Since a dividend is always equal to the product of its quotient and divisor, this definition of a factor is equivalent to the following: If an algebraic expression is exactly divisible by another, that divisor is a *factor* of the expression.

For example: $3 \times 2 = 6$ and the factors of 6 are 3 and 2; $3a^2b$ is the product of 3, a, a and b and these are the factors of $3a^2b$; the expression $a^3 - 3a^2b + 3ab^2 - b^3$ is exactly divisible by $a - b$ and $a - b$ is a factor of the given expression.

When the factors of an expression are found the expression is said to be *factored* or separated into factors, and the process of separation into factors is called *factoring* or *factorization.*

Since an expression (no matter how it actually originates) may be thought of as being formed as the product of its factors, factorization may be thought of as the reverse of multiplication and, therefore, akin to division. In fact, when one or more factors of an expression are known, other factors may be found by division.

23. Monomial Factors. A monomial factor of a polynomial is a monomial which is a factor of every term of the polynomial. Thus, 3 is seen at a glance to be a factor of every term of the trinomial $3a + 6ab - 27x^2$ and is, therefore, a factor of the trinomial. The trinomial is, therefore, the product of the factors 3 and $a + 2ab - 9x^2$ and it may be written $3(a + 2ab - 9x^2)$. When these factors are once known and the original expression is written as

$$3a + 6ab - 27x^2 = 3(a + 2ab - 9x^2)$$

it has been factored and we are said to have "taken out 3 as a factor."

Consider the expression

$$-2xy + 12ax - 10bx^2 - 2x.$$

Inspection of each term shows that it is exactly divisible by $-2x$. If the entire expression is divided through by $-2x$ and the rule of signs is remembered, the result is $y - 6a + 5bx + 1$. This result and $-2x$ are, therefore, factors of the original expression and by taking out the monomial factor $-2x$ we have

$$-2xy + 12ax - 10bx^2 - 2x = -2x(y - 6a + 5bx + 1).$$

Examining this result, it is seen that there is a minus sign before the parentheses and the monomial factor, and that the sign of every term inside the parentheses has been changed. Thus, in factorization the rule of signs for parentheses is applied whenever a negative factor is taken out. The result of factoring may be checked by multiplying the factors. The product should be the original expression.

From these examples it is seen that monomial factors may be found by inspection. A few more examples will illustrate the process sufficiently:

$$12ab - 4a^2b^2x + 32a^3by = 4ab(3 - abx + 8a^2y)$$

$$-72x^4y^2z - 36xy^2z + 84x^3y^3 = -12xy^2(6x^3z + 3z - 7x^2y)$$

$$3a^2bc^2d - 4ab^2cx^2 - 5abc^2y = abc(3acd - 4bx^2 - 5cy).$$

24. Polynomial Factors. If the binomial $(2a + 3x)$ is thought of as a single number and is used to multiply the expression $3b - 4c + 5y$, then according to the rule of article 16 there results

$$(2a + 3x)(3b - 4c + 5y) = (2a + 3x)3b - (2a + 3x)4c$$
$$+ (2a + 3x)5y = 6ab + 9bx - 8ac - 12cx + 10ay + 15xy.$$

If, therefore, we are given the expression

$$6ab + 9bx - 8ac - 12cx + 10ay + 15xy \qquad (9)$$

to factor, we can group the terms in parentheses as follows:

$$(6ab + 9bx) - (8ac + 12cx) + (10ay + 15xy),$$

and on examining the three binomials in parentheses, separately, it is seen at once that $3b$ is a monomial factor of the first expression, $4c$ is a monomial factor of the second, and $5y$ of the third. Taking out these monomial factors as in the preceding article, we get:

$$[3b(2a + 3x)] - [4c(2a + 3x)] + [5y(2a + 3x)].$$

This expression is a trinomial of two positive terms and one negative term and as seen by inspection the binomial $(2a + 3x)$ is a factor of

each term. Taking out this factor (considered as a single number) from each of the three terms as in the preceding article, the original expression (9), a polynomial of six terms, becomes

$$(2a + 3x)(3b - 4c + 5y). \tag{10}$$

The original six-term expression has been separated into two factors, a binomial and a trinomial.

In this case, we had the complete formation of the expression (9) as a product written out to serve as a guide in working backwards to the factored expression (10). Suppose, however, that we are given such an expression as (9) without knowing how it was formed; how shall we then proceed? Naturally, we should now first try to group the terms so that by inspection we could discover a monomial factor of each group such that when it is removed the remaining polynomial expression is the same in every group, this being the method used in factoring (9). If this can be done, this common polynomial is a factor of the original expression and can be taken out as illustrated in (10) above and explained in article 23.

Let us try to factor the following expression:

$$3a^2bc - 6acy^2 + abxy - 2xy^3 - 4abz^2 + 8y^2z^2. \tag{11}$$

To begin with, we find by inspection that $3ac$ is a monomial factor of the first two terms, xy is a factor of the third and fourth, and $4z^2$ of the last two. Therefore, let us group these terms in parentheses:

$$(3a^2bc - 6acy^2) + (abxy - 2xy^3) - (4abz^2 - 8y^2z^2).$$

Taking out the monomial factor in each group, this becomes

$$3ac(ab - 2y^2) + xy(ab - 2y^2) - 4z^2(ab - 2y^2).$$

Each of the three product terms of this expression has as a factor the binomial $(ab - 2y^2)$. Taking out this factor from the three terms we have, finally:

$$(ab - 2y^2)(3ac + xy - 4z^2), \tag{12}$$

and this is the factored form of the given expression (11).

The proper method of grouping the terms of an expression to be factored is not always so obvious as in the case of the expressions (9) and (10). For example, let us try the expression

$$12abcxy - 14a^2z - 36x^3y^2 + 35a^2bxz \tag{13}$$
$$+ 6xy - 28a^3bcz - 15bx^2y + 84a^2x^2yz.$$

At first there seems no way to group the terms in pairs or fours but careful examination shows that the first and sixth have the common factor $4abc$, and, similarly, other pairs of terms are found to contain common factors. Let us try the following rearrangement of the terms:

$$12abcxy - 28a^3bcz - 36x^3y^2 + 84a^2x^2yz$$
$$- 15bx^2y + 35a^2bxz + 6xy - 14a^2z.$$

It is now seen that the first two terms have the common factor $4abc$, the next two have the factor $12x^2y$, the next two the factor $5bx$, and the last two the factor 2.

Grouping the terms in pairs in parentheses we have:

$$(12abcxy - 28a^3bcz) - (36x^3y^2 - 84a^2x^2yz)$$
$$- (15bx^2y - 35a^2bxz) + (6xy - 14a^2z).$$

Taking out the monomial factors in each group we have

$$4abc(3xy - 7a^2z) - 12x^2y(3xy - 7a^2z)$$
$$- 5bx(3xy - 7a^2z) + 2(3xy - 7a^2z).$$

Taking out the now common binomial factor $(3xy - 7a^2z)$ from each term in this expression we get, finally:

$$(3xy - 7a^2z)(4abc - 12x^2y - 5bx + 2) \tag{14}$$

which is the factored form of (13).

The expression (13) may also be grouped as follows:

$$(12abcxy - 36x^3y^2 - 15bx^2y + 6xy)$$
$$- (28a^3bcz - 84a^2x^2yz - 35a^2bxz + 14a^2z),$$

and in this form, inspection reveals that the first group has the common monomial factor $3xy$ and the second, $7a^2z$. Taking out these factors the expression becomes:

$$3xy(4abc - 12x^2y - 5bx + 2) - 7a^2z(4abc - 12x^2y - 5bx + 2),$$

and the two expressions in parentheses are the same. Taking out this four term expression as a common factor of the two terms, therefore, we have:

$$(3xy - 7a^2z)(4abc - 12x^2y - 5bx + 2),$$

which is the same as (14).

As a final example let us factor

$$ax + by + cz + ay + bz + cx + az + bx + cy. \tag{15}$$

Rearranging and grouping, this becomes:

$$(ax + ay + az) + (bx + by + bz) + (cx + cy + cz).$$

Removing the common factor from each group we get

$$a(x + y + z) + b(x + y + z) + c(x + y + z),$$

so that we have, finally,

$$(a + b + c)(x + y + z) \tag{16}$$

as the factored form of (15).

The expressions (9), (11), (13), (15) and their factored forms (10), (12), (14), (16) indicate something of the general procedure in finding binomial, trinomial and other polynomial factors. Many expressions are more difficult to group and many others cannot be factored at all. Also, in many cases, after polynomial factors have been found as in (10), (12), (14), (16) above, it will be found that one or more of these factors may in turn be factored. If such is the case, however, the factor may be taken as a new expression and factored in the same way as those illustrated above. The final factors are then all those which have been found and the original expression is equal to their product.

25. Factors of Trinomials. In the exercises of article 21, it is noticeable that many products of two binomials are trinomials. Thus, $(x + 3)(x + 4) = x^2 + 7x + 12$; $(x + 3)(x - 4) = x^2 - x - 12$. It is often necessary to find the factors of such trinomials.

Let us examine the product of two binomials more closely:

$$(x + 5)(x + 2) = x^2 + 7x + 10$$

$$(y - 3)(y + 4) = y^2 + y - 12$$

$$(2a + 1)(3a - 5) = 6a^2 - 7a - 5$$

The following observations may be made:

1. The first term in each binomial factor is a factor of the first term in the trinomial.
2. The second term in each binomial factor is a factor of the third term in the trinomial.
3. The algebraic sum of the products formed by the terms indicated by the arrows is equal to the middle term of the trinomial.

For example, in the products above, $+5x + 2x = 7x$

$$-3y + 4y = +y$$

$$+3a - 10a = -7a$$

We can see why this is so if we examine the following multiplication:

$$\begin{array}{rr} & x+5 \\ & x+2 \\ \hline & 2x+10 \\ x^2+5x & \\ \hline x^2+7x & +10 \end{array}$$

Product of x and x Sum of products Product of 5 and 2

$$\begin{array}{rr} 2(x) = & 2x \\ 5(x) = & 5x \\ \hline & +7x \end{array}$$

Thus, $x^2 + 5x + 10 = (x + 5)(x + 2)$

In seeking the factors which will yield the correct middle term, we had to find two terms whose algebraic sum is $+7x$ and whose product is $+10$.

This method will be explored further in the following illustrative examples.

EXAMPLE: Factor $y^2 + 3y - 10$

The factors of this trinomial will be two binomials. $(\quad)(\quad)$.

The factors which yield y^2 are y and y. Thus, we have $(y \quad)(y \quad)$. The factors which yield -10 are $+10$ and -1, -10 and $+1$, $+5$ and -2, -5 and $+2$. However, we must find a pair whose algebraic sum will yield $+3$, the middle term.

These factors are $+5$ and -2.

Thus, $y^2 + 3y - 10 = (y + 5)(y - 2)$.

To check this result, we see that $(y + 5)(y - 2)$, the algebraic sum of the products indicated by the arrows, is $3y$.

EXAMPLE: Factor $2x^2 - 5x + 3$

The factors of this trinomial will be two binomials $(\quad)(\quad)$.

The factors which yield $2x^2$ are $2x$ and x. Thus we have $(2x \quad)(x \quad)$.

The factors which yield $+3$ are $+1$ and $+3$, or -1 and

-3. We must arrange these factors in the spaces provided above so that the algebraic sum of the products indicated by the arrows is $-5x$.

Thus, $2x^2 - 5x - 3 = (2x - 3)(x - 1)$.

We may check this result by finding the algebraic sum of the products indicated by the arrows.

EXAMPLE: Factor $10x^2 - 7x - 12$.

The factors of this trinomial will be two binomials $(\quad)(\quad)$.

The factors which yield $10x^2$, are $10x$ and x, $-10x$ and $-x$, $5x$ and $2x$, $-5x$ and $-2x$. By trial, we discover that $5x$ and $2x$ are the desired factors. Thus we have $(5x \quad)(2x \quad)$.

The number -12 may be factored in several ways. For example, we may have $+3$ and -4, -6 and $+2$, etc. Again, by trial, we discover that $+4$ and -3 are the desired factors.

Thus, $10x^2 - 7x - 12 = (5x + 4)(2x - 3)$.

This result should be checked by finding the algebraic sum of the products indicated by the arrows.

It should be noted that a considerable number of trials may be necessary in some cases. However, if the trinomial is factorable, there should be no real difficulty in identifying the factors.

26. Factors of Some Special Expressions. In article 17, we have considered some products of binomials which are of frequent occurrence in the study and application of mathematics. They will be re-stated here in the form of factored expressions.

Thus, from the expressions (1), (3) and (4) of that article we can write

$$\left.\begin{aligned} x^2 + 2xy + y^2 &= (x+y)(x+y) = (x+y)^2 \\ x^2 - 2xy + y^2 &= (x-y)(x-y) = (x-y)^2 \\ x^2 - y^2 &= (x+y)(x-y) \end{aligned}\right\} \tag{22}$$

There are several other expressions which are of frequent occurrence and are at once factorable. One of these is the difference of two cubes, $x^3 - y^3$. In order to factor this, we recall one of our definitions of a factor of an expression, as an exact divisor of that expression. Comparing the expression $x^3 - y^3$ with the last of (22) and noting

that $x - y$ is a factor of $x^2 - y^2$ we are led to try $x - y$ as a factor of $x^3 - y^3$. So let us divide $x^3 - y^3$ by $x - y$.

$$\begin{array}{r|l}
 & x^2 + xy + y^2 \\
\hline
x - y & x^3 - y^3 \\
 & x^3 - x^2y \\
\hline
 & \quad x^2y - y^3 \\
 & \quad x^2y - xy^2 \\
\hline
 & \qquad xy^2 - y^3 \\
 & \qquad xy^2 - y^3 \\
\hline
\end{array}$$

The division is exact and the quotient is $x^2 + xy + y^2$. This trinomial quotient and the binomial divisor $x - y$ are, therefore, factors of $x^3 - y^3$. We can write, therefore,

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2), \tag{23}$$

and in this expression, as in (22), x and y represent any numbers or expressions whatever.

Another such expression is the sum of two cubes, $x^3 + y^3$. Comparison of this with (23) suggests that $x + y$ may be a factor and that the other factor may be $x^2 - xy + y^2$. If this is the case we should have

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2), \tag{24}$$

which may be checked by finding the product of the assumed factors or by dividing $x^3 + y^3$ by either of them. Let us multiply:

$$\begin{array}{llll}
x^2 & - xy & + y^2 & \\
x & + y & & \\
\hline
x^3 & - x^2y & + xy^2 & \\
 & x^2y & - xy^2 & + y^3 \\
\hline
x^3 & & & + y^3
\end{array}$$

The product is $x^3 + y^3$ and the factors are, therefore, those given in (24).

In (24) as in (22) and (23), x and y may be any numbers, single symbols, products, powers, binomials, etc.

27. Remarks on Factorization. It is obvious that while *any* expressions may be multiplied to form a product, not every expression is a product of factors. There are many expressions which are not

factorable. Even when an expression is factorable, its factorization may be a matter of some difficulty. Of the many forms of algebraic expressions which are factorable only a few can be factored by definite rules. Thus, the method for taking out monomial factors, as in article 23, is purely a matter of inspection and recognition of the factors, and the finding of polynomial factors, as in article 24, also depends on finding groups of terms with recognizable common monomial factors. As was seen in one of the examples of article 24, the grouping of terms may be carried out in different ways and there is no simple general rule by which polynomial factors can readily be found even when they exist.

The rules given in article 25 apply to certain particular forms of trinomials, but even these cannot be used blindly; they require some ingenuity and practice. It so happens, however, that the types of expressions to which these rules apply are of considerable importance and frequent occurrence. Exercises are given in the next article which represent all the forms studied in this chapter, and those studied in articles 25 and 26 will be useful to us later in this book.

28. Exercises.

Take out common monomial factors in the following:

1. $a^3 - ax$; $x^3 - x^2$; $4ab + 2bc$; $3a^2x + 9ab - 12abx$.
2. $8x - 2x^2$; $50x + 5a^3x^2$; $54 - 81y$; $3z^3 - z^2 + z$.
3. $38a^3x^5 + 57a^4x^2$; $5x^5 - 10a^2x^3 - 15a^3x^3$.

Separate the following into binomial factors:

4. $x^2 - ax + bx - ab$.
5. $6x^2 - 9ax + 4bx - 6ab$.
6. $12a^2 + 4ab - 3ax^2 - bx^2$.
7. $3x^3 - 5x^2 + 3x - 5$.

Factor the following trinomials:

8. $x^2 + 11x + 24$.
9. $x^2 - 10x + 24$.
10. $y^2 - 18y + 81$.
11. $z^2 + 12z + 35$.
12. $x^2 - x - 2$.
13. $x^2 + x - 2$.
14. $y^2 + 18y - 115$.
15. $x^4 - 14x^2 - 51$.
16. $2x^2 + 3x + 1$.
17. $3x^2 + 8x + 4$.

18. $6y^2 + 7y - 3$.

19. Show that the factors of $x^2 + 2xy + y^2 - a^2$ can be $(x + y + a)(x + y - a)$.

20. Show that the factors of $x^2 - 6ax + 9a^2 - 16b^2$ can be $(x - 3a + 4b) \times (x - 3a - 4b)$.

21. Show that the factors of $x^2 - a^2 - 2ab - b^2$ can be $(x + a + b)(x - a - b)$.

22. Show that the factors of $a^2 - 4ab + 4b^2 - 9a^2c^2$ can be $(a - 2b + 3ac) \times (a - 2b - 3ac)$.

23. Show that the factors of $x^4 + 16x^2 + 256$ can be $(x^2 + 4x + 16) \times (x^2 - 4x + 16)$.

24. Show that the factors of $81a^4 + 9a^2b^2 + b^4$ can be $(9a^2 + 3ab + b^2) \times (9a^2 - 3ab + b^2)$.

25. Show that the factors of $9a^2 - 6a + 1 - x^2 - 8bx - 16b^2$ can be $(3a - 1 + x + 4b)(3a - 1 - x - 4b)$.

[illegible]

19. Show that the [illegible]

20. Show that the factors of [illegible]

21. Show that the [illegible]

22. Show that the [illegible]

23. Show that the equation [illegible]

24. Show that the factors of [illegible]

25. Show that the factors of [illegible]

Part 2

FUNDAMENTAL PRINCIPLES

Chapter 5

POWERS AND ROOTS

29. Powers. In articles 2 and 3, we have already learned what is meant by a *power* of a number (numerical or algebraic) and how powers are indicated by exponents, and in article 17, we studied a few of the powers of certain binomials. As seen in those pages, a power of any number is the product obtained by using that number as a factor several times, the number of times the same factor is used being the power to which it is *raised*, and this power being indicated by the *exponent* of the number. Thus, instead of writing $4 \times 4 \times 4 \times 4 \times 4$ to indicate that the number 4 is to be used as a factor 5 times we write 4^5 and this is the fifth power of 4, 5 being the exponent. By carrying out the multiplication it is found that the actual *value* of the fifth power of 4 is 1024. This is written $4^5 = 1024$. Similarly, for any algebraic number a, $a \times a \times a \times a$ is written a^4, and so for any other power; the *value* of a^4 is not known, however, unless the numerical value of the symbol a is specified. Thus, if $a = 3$, $a^4 = 81$; if $a = 5$, $a^4 = 625$; etc.

In order to raise the algebraic monomial $2ax$ to any power, say the third, we write $(2ax)^3 = 2ax \times 2ax \times 2ax$ and by the rules of multiplication this is $2 \times 2 \times 2 \times a \times a \times a \times x \times x \times x = 2^3 \times a^3 \times x^3 = 8a^3x^3$. That is, *to raise the product of several factors to any power, raise each factor to that power.* Thus, $(3xy)^2 = 9x^2y^2$, $(4bcy)^4 = 256b^4c^4y^4$, etc.

In the same way, we may raise a polynomial to any power. As already seen $(x + y)(x + y)$ is written $(x + y)^2$ and by multiplication we found $(x + y)^2 = x^2 + y^2 + 2xy$. It is especially to be noted that $(x + y)^2$ is NOT equal to $(x^2 + y^2)$ but to $(x^2 + y^2) + 2xy$. This means that we can NOT square a binomial simply by squaring each of its terms; the binomial must be multiplied by itself and the full multiplication gives three terms instead of two. Similarly, we cannot say that $(x + y)^3$ is equal to $(x^3 + y^3)$ but must remember that $(x + y)^3$ means $(x + y)(x + y)(x + y)$, and when this multiplication is carried out as in article 17, it is found that $(x + y)^3 = (x^3 + y^3) + 3x^2y + 3xy^2$.

Similarly $(a + b + c)^2$ is not $(a^2 + b^2 + c^2)$ but $(a^2 + b^2 + c^2) + 2ab + 2bc + 2ca$, as may be verified by carrying out the multiplication $(a + b + c)(a + b + c)$.

In the same way, we would indicate the square of the polynomial $2ab + 4xy - 6yz^2$, which is the product $(2ab + 4xv - 6yz^2) \times (2ab + 4xy - 6yz^2)$, by writing $(2ab + 4xy - 6yz^2)^2$ and the square of this trinomial can be written out in full by performing the indicated multiplication.

Let us find the square of the negative number -2. This is written $(-2)^2$ and is equal to $(-2)\cdot(-2) = +4$, by the rule of signs for multiplication. That is

$$(-2)^2 = +4.$$

Similarly, the cube of -2 is $(-2) \times (-2) \times (-2) = -8$, or

$$(-2)^3 = -8.$$

By repeating the same multiplications we find that

$$(-2)^4 = +16,$$
$$(-2)^5 = -32,$$
$$(-2)^6 = +64,$$
$$(-2)^7 = -128, \text{ etc.}$$

Now, 2, 4, 6, etc., are even numbers and 3, 5, 7, etc., are odd numbers. So the 2nd, 4th, 6th, etc., powers are called *even powers* and the 3rd, 5th, 7th, etc., powers are called *odd powers.* Examining these results we see that the numbers which are the even powers of -2 are positive and those which are the odd powers of -2 are negative. In the same way, we find for *any* number a. $(-a)^2 = a^2$, $(-a)^3 = -a^3$, $(-a)^4 = a^4$, $(-a)^5 = -a^5$, etc. That is, in general, for all numbers, *even powers of a negative number are positive, odd powers of a negative number are negative.* All powers of positive numbers are, however, positive since no negative factor enters into the multiplication.

30. Roots. The operation which is the inverse of raising a number to a power is called finding a *root* of a number. Thus, 25 is the second power of 5, and 5 is the second root of 25; 216 is the third power of 6, and 6 is the third root of 216. Since the second power is called the square and the third power the cube (for the reason stated in article 2) the second root is called the *square root* and the third root the *cube root.*

Thus, we say that the square root of 25 is 5 and the cube root of 216 is 6.

The symbol used to indicate a root of a number is $\sqrt{\ }$ with the number written under the "bar" and the figure indicating the root written in the "hook." Thus, we write the "cube root of 216" as $\sqrt[3]{216}$ and the "square root of 25" may be written $\sqrt[2]{25}$. This sign is called a *radical* and the small figure indicating the root is called the root *index*. As there is no integral root of a lower index than the second, beside the number itself which is its own first root, the index is not written for the square root. Thus, the square root of 25 is written simply $\sqrt{25}$. For all other roots, however, the root index must be written. Thus, the fifth root of 243 is indicated by $\sqrt[5]{243}$; similarly, $\sqrt{a}$ is the square root of the number a; $\sqrt[3]{a+b}$ is the cube root of the single number which is the sum of a and b; $\sqrt[4]{7bc}$ is the fourth root of the *product* of the numbers b, c and 7.

Since the square root of 25 is 5 we write $\sqrt{25} = 5$. Similarly, $\sqrt[3]{216} = 6$, $\sqrt[5]{243} = 3$, etc. If it is known that a certain number is a particular power of a certain other number, then it is, of course, also known that that other number is the corresponding root of the first number. Thus, since it is known by multiplication that $3^5 = 243$ it is also known at once that $\sqrt[5]{243} = 3$. If, however, it is not known what original number was raised to a certain power to produce any particular number, then the corresponding root of that number is not known but must be found by special methods. The finding of any root of a number is referred to as *taking* or *extracting* the root. Thus, to find $\sqrt[3]{216}$ we are said to "take the cube root" or to "extract the cube root" of 216. This process is also called *evolution* and the corresponding process of raising numbers to powers is called *involution*.

While a number is raised to a power by multiplication, it is multiplication of the number by *itself* and NOT by the exponent of the power. Similarly, a root of a number is NOT found by dividing the number by the index of the root. Thus, 10^2 is 10×10 and not 10×2, and $\sqrt[4]{16}$ is not $16 \div 4$, for $16 \div 4$ is 4 while we know that $\sqrt[4]{16} = 2$ because $2^4 = 16$. Similarly, $\sqrt{a}$ is not $a/2$, etc.

Special methods are given in arithmetic for finding the square and cube roots or ordinary numbers, whether the root is exact or not, but

no regular rules are given for extracting higher roots. The same methods are given in algebra for finding the square and cube roots of algebraic polynomials. In a later chapter we shall find a very simple method of extracting not only the square and cube roots but *any* root of an ordinary number. We shall presently give the rule for taking the square root of a polynomial.

31. Some Important Properties of Square and Cube Roots. We have seen that for any binomial such as $a + b$ or $a - b$ we cannot write $(a + b)^2$ equal to $a^2 + b^2$ or $(a - b)^2$ equal to $a^2 - b^2$. Similarly, we cannot write $\sqrt{a + b}$ equal to $\sqrt{a} + \sqrt{b}$ or $\sqrt[3]{a - b}$ equal to $\sqrt[3]{a} - \sqrt[3]{b}$. This is seen at once if we square the binomial $\sqrt{a} + \sqrt{b}$. According to the rule for the square of a binomial $(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a})^2 + (\sqrt{b})^2 + 2(\sqrt{a} \times \sqrt{b})$. But since the square of the square root of a number is the number itself $(\sqrt{a})^2 = a$ and $(\sqrt{b})^2 = b$. Therefore $(\sqrt{a} + \sqrt{b})^2 = a + b + 2(\sqrt{a} \times \sqrt{b})$, and since we thus find that $(\sqrt{a} + \sqrt{b})^2$ is not $a + b$, then $\sqrt{a + b}$ is not $\sqrt{a} + \sqrt{b}$.

The same holds for roots of any other polynomials. Thus $\sqrt{9 + 16}$ is not $\sqrt{9} + \sqrt{16}$. For, $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$, while $\sqrt{9 + 16} = \sqrt{25} = 5$, which is not the same. Similarly, $\sqrt{a^2 + b^2}$ is not $a + b$, for $(a + b)^2$ is not $a^2 + b^2$; $\sqrt[3]{x^3 + y^3}$ is not $x + y$ for $(x + y)^3$ is not $x^3 + y^3$; $\sqrt[3]{2ax - b^2 + 3by}$ is not $\sqrt[3]{2ax} - \sqrt[3]{b^2} + \sqrt[3]{3by}$; etc. That is, *a root of a polynomial is* NOT *found by taking the corresponding root of each of the separate terms.* This is a very important principle and must be remembered. We shall often have to deal with square and cube roots and the mistake of taking the roots of the separate terms under the radical must not be made.

We saw in article 29 that an odd power of a negative number is always negative. Therefore, *an odd root of a negative number is negative.* For example, since $(-2)^3 = -8$, then $\sqrt[3]{-8} = -2$; $\sqrt[5]{-243} = -3$, $\sqrt[3]{-27a^3} = -3a$, $\sqrt[7]{-x^7y^7} = -xy$, etc. In particular, since the cube root is an odd root, *the cube root of any negative number or algebraic expression is negative.*

Also, since an even power of a negative quantity is positive, and all powers of a positive number are positive, odd roots of positive numbers are always positive, but *even* roots of positive numbers may be *positive or negative.*

In particular, the square root of a positive number, being an even root, may be positive or negative. Thus, since $(+3)^2 = +9$ and $(-3)^2 = +9$, $\sqrt{9} = +3$ or $\sqrt{9} = -3$. This is usually written $\sqrt{9} = \pm 3$. Similarly, since $(+a)^2 = a^2$ and $(-a)^2 = a^2$, $\sqrt{a^2} = \pm a$, $\sqrt{(a+b)^2} = \pm(a+b)$, etc. This means that to be strictly correct we should in taking square roots of positive numbers write both plus and minus ($\pm$) before the root. In many cases, no sign is written at all. In such cases the root is taken as plus, but in certain cases we shall find that both signs are necessary and that it is also necessary to distinguish between them.

There is one more important property which certain square roots possess. Suppose we square (-2); as above we get $+4$. Also, $(+2)^2 = +4$. Similarly, also as seen before, $(+a)^2 = +a^2$ and $(-a)^2 = +a^2$, and so for any number. That is, *no* real *number*, positive or negative, *produces a negative number when squared.* This means that according to our usual rules and definitions, *negative numbers have no square roots.* Therefore, a number such as $\sqrt{-9}$, $\sqrt{-a^2}$, $\sqrt{-16x^2y}$, etc., are not real numbers. Such numbers are called *imaginary numbers.* Strange as it may seem, such numbers are of great importance in higher mathematics and are much used in electrical engineering calculations. In a later chapter we shall investigate some of the properties of the so-called imaginary numbers.

32. Square Root of a Polynomial. We know that $(a + b)^2 = a^2 + 2ab + b^2$, and $(x - y)^2 = x^2 - 2xy + y^2$. Therefore, $\sqrt{a^2 + 2ab + b^2} = a + b$, and $\sqrt{x^2 - 2xy + y^2} = x - y$. But, suppose we did not know that the square of $c + b$ is $a^2 + 2ab + b^2$, or were asked to find the square root of some other polynomial which was not known in advance; how should we proceed to find the root? It has been stated that there is a method for finding such square roots. We give here a brief account of the method.

Let us consider the polynomial $a^2 + 2ab + b^2$. Since it is the square of some other polynomial, that is, the product formed by using the same polynomial twice as a factor, we know that the term a^2 comes from a multiplication of a by a, that is, from squaring the term a. Therefore, the first step in the reverse process is to take the square root of a^2 which is a.

From our study of factorization in article 25 we know that the term $2ab$ is the sum of cross products of end terms. To determine these

terms we try a certain one as a *trial divisor* or factor. To determine this trial divisor double that part of the root already found, namely a; this gives $2a$. Divide this into the first of the remaining terms $2ab + b^2$; it goes b times. Write this quotient as a term of the required root, and also add it to the trial divisor $2a$, giving $2a + b$ for a complete divisor.

Next, multiply the complete divisor $2a + b$ by the second term b of the required root and write the product $2ab + b^2$ under the remainder terms, $2ab + b^2$, of the original polynomial. Finally, subtract this product from the terms above it. The final remainder is zero, the process is complete, the "root-quotient" is exact and the required square root is $a + b$.

The work is carried out as follows:

$$\begin{array}{rl} & \cancel{a^2} + 2ab + b^2(a + b \\ & \cancel{a^2} \\ \hline 2a + b) & 2ab + b^2 \\ & \underline{2ab + b^2} \end{array}$$

Using this method, let us take the square root of the trinomial $4x^2 - 12xy + 9y^2$.

$$\begin{array}{rl} & 4\cancel{x^2} - 12xy + 9y^2(2x - 3y \\ & 4\cancel{x^2} \\ \hline 4x - 3y) & -12xy + 9y^2 \\ & \underline{-12xy + 9y^2} \end{array}$$

Therefore,

$$\sqrt{4x^2 - 12xy + 9y^2} = 2x - 3y.$$

By tracing out the steps of the process used in these examples, we find that they can be repeated for polynomials of more terms and may be stated as follows:

RULE FOR SQUARE ROOT OF A POLYNOMIAL: (1) *Arrange th terms in descending order of one of the letters.*

(2) *Take the square root of the first term and write this root as the first term of the required root. Square this term, subtract this square from the given polynomial and write down the remainder as a dividend.*

(3) *For a trial divisor, double the root already found and divide this result into the first term of the above remainder-dividend. Write the quotient as the next term of the required root, and also add it to the trial divisor to form the complete divisor.*

(4) *Multiply the complete divisor by the last term of the root and subtract the product from the dividend. Write down the remainder as a new dividend.*

(5) *For a trial divisor double all the terms of the root already found and proceed again as in steps* (3) *and* (4).

(6) *Continue this process until there is no remainder. The complete square root is the sum of all the separate terms found.*

(7) *The usual rules of signs apply in all the various steps.*

As a final illustration, we apply this rule to find the square root of the polynomial $a^2 + 2ab + 2ac + 2bc + b^2 + c^2$.

$$
\begin{array}{rl}
 & \cancel{a^2} + 2ab + 2ac + 2bc + b^2 + c^2(a + b + c \\
 & \cancel{a^2} \\
\hline
2a + b) & 2\cancel{a}b + 2ac \\
 & 2\cancel{a}b + b^2 \\
\hline
2a + 2b + c) & 2ac - \cancel{b^2} + 2bc + \cancel{b^2} + c^2 \\
 & 2ac \qquad + 2bc \qquad + c^2 \\
\hline
\end{array}
$$

The rule for finding the cube root of polynomials is not often needed and is rather long and complicated. It is, therefore, not given here.

33. Exercises.

Raise the following expressions to the powers indicated:

1. $(2x)^2$; $(3a)^3$; $(4xy)^2$; $(4abc)^3$; $(2xyz)^4$.
2. $(y - z)^2$; $(z - y)^2$; $(2x + a)^2$; $(a - 3b)^2$.
3. $(a + b + c)^2$; $(x - y - z)^2$; $(a - 2b + 3c)^2$.
4. $(x + y + z)^3$.
5. $(a - 2b + 3c)^3$.
6. $(3x + 2y + z)^3$.

Extract the indicated roots of the following expressions:

7. $\sqrt{36a^2}$; $\sqrt{16x^4y^4}$; $\sqrt[3]{27a^3x^3}$; $\sqrt[4]{81b^4z^8}$.
8. $\sqrt{4(x - a)^2}$; $\sqrt[3]{8(a + b)^3}$; $\sqrt{4x^2(2x - 3y)^2}$.
9. $\sqrt[3]{(a + b)^3 \times (c - d)^6}$.

Find the square roots of the following polynomials:

10. $x^2 - 10xy + 25y^2$.
11. $4x^4 - 12x^3 + 29x^2 - 30x + 25$.
12. $6ab^2c - 4a^2bc + a^2b^2 + 4a^2c^2 + 9b^2c^2 - 12abc^2$.

Chapter 6

LAWS OF EXPONENTS

34. Introductory Remarks. In mathematics a *law* is a rule, or a relation among numbers or symbols for numbers, which always holds good under the same conditions as those for which it is stated. Thus, the rules of signs which we have found to fit our definitions of positive and negative numbers and which always hold good under the conditions under which those numbers may be used, may be called the *laws of signs*. Similarly, the *laws of exponents* are the relations which exist among powers and exponents as these are defined in articles 2 and 3.

Although we have defined positive and negative numbers, and zero is the symbol or number which is neither positive nor negative, we have so far only used positive numbers as exponents. In this chapter we shall find that either positive or negative numbers or zero may be used as exponents and shall study their meanings and laws. We begin with the positive exponents with which we are already acquainted.

35. Positive Exponents. We have seen that, if a represents any number whatever, $a^2 = a \times a$, $a^3 = a \times a \times a$, $a^4 = a \times a \times a \times a$, etc. Similarly, a^m is the product obtained by using the number a as a factor m times. That is, $a^m = a \times a \times a \times a \times a \ldots$, m times, and, similarly, for a^n or any other exponent represented by a symbol or a number.

Now, as already seen, $a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a \times a \times a \times a \times a = a^5 = a^{2+3}$. Similarly, $a^4 \times a^6 = a^{4+6} = a^{10}$, and for any positive exponents m, n, etc.,

$$a^m \times a^n = a^{m+n}. \tag{1}$$

This is a very important law and one which we have already used in Chapter 3.

Consider now the quotient $a^6 \div a^4$, or, as it is more conveniently written, $\frac{a^6}{a^4}$. This means $\frac{a \times a \times a \times a \times a \times a}{a \times a \times a \times a} = a \times a = a^2 = a^{6-4}$.

That is, $\frac{a^6}{a^4} = a^{6-4}$, which means that the number a is used 6 times as a multiplier and 4 times as a divisor, with the net result that a is

used twice as a factor. Similarly, $\dfrac{a^7}{a^3} = a^{7-3} = a^4$, and, in general, for any positive exponents m, n,

$$\frac{a^m}{a^n} = a^{m-n}. \tag{2}$$

This is another relation which is very important and one which we have already made use of in Chapter 3.

The relations (1) and (2) are the fundamental laws of exponents and from them all others are derived. It is seen at once that these two are nothing more than a direct consequence of the definition of an exponent.

Suppose we wish to raise the number a to the second power, and then take this result as a new number and raise it to the third power. This is indicated by writing $(a^2)^3$. But the cube of a^2 is $(a^2) \times (a^2) \times (a^2)$ and according to (1) above this is equal to $a^{2+2+2} = a^6 = a^{3\times2}$. That is, $(a^2)^3 = a^{3\times2}$. Similarly, $(a^3)^4 = a^{4\times3} = a^{12}$, and, in general,

$$(a^n)^m = a^{mn}. \tag{3}$$

The relations (1), (2) and (3) will be much used hereafter and should be thoroughly understood and remembered.

Next let us consider the expression $(ab)^3$. This is equal to

$$\begin{aligned}(ab) \times (ab) \times (ab) &= a \times b \times a \times b \times a \times b \\ &= (a \times a \times a) \times (b \times b \times b) = a^3 \times b^3.\end{aligned}$$

That is, $a(b)^3 = a^3b^3$. Similarly, for any positive exponent m,

$$(ab)^m = a^m b^m. \tag{4}$$

In the same way $\left(\dfrac{a}{b}\right)^3 = \dfrac{a}{b} \times \dfrac{a}{b} \times \dfrac{a}{b}$ and according to the law of multiplication of fractions this is equal to $\dfrac{a \times a \times a}{b \times b \times b} = \dfrac{a^3}{b^3}\cdot$ That is, $\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$, and, in general,

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\cdot \tag{5}$$

In (4) and (5), m may be any real number. The relations (4) and (5) may, therefore, be stated in words by saying that: A power of a product is the product of the powers, and a power of a quotient is the

quotient of the powers, the separate powers applying to the factors, numerator and denominator. It is to be remembered that the same is NOT true of a sum or difference, as we have already seen in the preceding chapter.

36. Zero Exponent. Suppose that in formula (2) above the exponents m and n are the same, that is, $n = m$. We then have $a^m/a^m = a^{m-m} = a^0$. But $a^m \div a^m = 1$, or $a^m/a^m = 1$. Therefore, since also $a^m/a^m = a^0$, it must be that

$$a^0 = 1. \tag{6}$$

Here a represents *any* number except zero. This result, therefore, states that if any number except zero is raised to the zero power the result is 1. We cannot be said to have *proved* that this is so, but having set up our definitions of exponents and their meaning in multiplication and division, the relation (6) then specifies, *in accordance with the definitions*, what we *mean* by the zero power of a number. That is, (6) is in reality the *definition* of the zero power.

The beginning student frequently thinks that when a number is raised to the zero power the result is zero, but it must be remembered that an exponent does not indicate multiplication by the number or symbol which is the exponent, but multiplication by the number which is being raised to the power, and multiplication repeated a number of times which is indicated by the exponent. Thus a^2 indicates multiplication by the number a twice. Similarly, a^0 would mean multiplication by a zero times, that is no multiplication by a at all. But if we do not multiply by a, that is if we multiply by a^0, the quantity thought of as being multiplied is not changed at all, which means simply that we have done the same as multiply it by 1.

If in any algebraic expression the zero power of any number occurs it may always be replaced by the number 1. Thus, $4^0 = 1$, $275^0 = 1$, $(3ab^2)^0 = 1$, $4xy + 10b^0 = 4xy + 10$, $8az - (7bx)^0 = 8az - 1$, etc.

37. Negative Exponents. If in the relation (2) we put $m = 0$ it becomes

$$\frac{a^0}{a^n} = a^{0-n} = a^{-n}.$$

But $a^0 = 1$ and, therefore, $\frac{a^0}{a^n} = \frac{1}{a^n}\cdot$ Hence, $\frac{1}{a^n} = a^{-n}$, or

$$a^{-n} = \frac{1}{a^n}\cdot \tag{7}$$

This states that raising of a number to a negative power is the same as division by the number raised to the *positive* of that power, or in other words, if any number or symbol appears as a factor in the numerator of a fraction with a negative exponent it may be shifted to the denominator of the fraction by changing the sign of the exponent to plus. Thus, $2x^{-3} = \frac{2}{x^3}$. By inverting both sides of the equation (7) we get also

$$\frac{1}{a^{-n}} = a^n, \tag{7a}$$

which means that we may also shift a factor with a negative exponent from denominator to numerator by changing the sign of the exponent.

Relations (7) and (7a) taken together mean that we may shift any factor with positive or negative exponent from numerator to denominator or vice versa by changing the sign of the exponent. As examples of the application of this principle: $\frac{3a}{b^{-2}} = 3ab^2$; $\frac{a^{-3}}{b^{-2}} = \frac{b^2}{a^3}$; $\frac{3x^2y^{-1}}{2^{-2}x^3} = \frac{3 \times 2^2}{xy} = \frac{12}{xy}$; $\frac{(3ax + by^{-1})}{(a^{-2} + b)} = \frac{\left(3ax + \frac{b}{y}\right)}{\left(\frac{1}{a^2} + b\right)}$, etc. Also, in general, multiplication by a number with a negative exponent means division by that number with the same exponent positive, and vice versa. Thus, $x^3 \times x^{-2} = x^3 \div x^{+2} = x$, $b^4 \div b^{-2} = b^4 \times b^{+2} = b^6$, etc. This shows that negative exponents add as do positive exponents in the formula (1), the addition being algebraic. Thus, $c^4 \times c^{-3} = c^{4-3} = c^1 = c$, $y^2/y^{-3} = y^{2-(-3)} = y^{2+3} = y^5$, etc.

38. Exercises.

Perform the indicated operations in each of the following:

1. $2a^2b \times 3ab^2$; $-4x^3y^4 \times 7y^3xz$; $x^{1/2} \cdot bx^{1/2}$.
2. $12a^4b^3/6a^2b$; $8x^3y^3z^3 \div (-4xyz)$; $36x^{3/2}y^{2/3} \div 12x^{1/2}y^{1/3}$.
3. $(4x^3)^2$; $(2ab^{1/2})^2$; $(x^{2/3}y^{4/3})^3$; $(a^{2/3}c^{4/3})^{3/2}$.

Express the following with positive exponents:

4. $2x^{-1/4}$; $3a^{-2/3}$; $\frac{1}{5}x^{-1/2}$; $4x^{-2}a^3$; $3 \div a^{-2}$; $\frac{1}{4}a^{-2}$.
5. $\frac{3a^{-3}x^2}{5y^2c^{-4}}$; $\frac{x^ay^{-b}}{x^{-a}y^b}$; $2x^{1/2} \times 3x^{-1}$; $a^{-2}x^{-1} \div 3x$.

6. Show that $[(x - y)^{-3}]^n \div [(x + y)^n]^3 = \dfrac{1}{(x^2 - y^2)^{3n}}$.

7. Show that $\left[\dfrac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}} \cdot \left(\dfrac{b^{1/4}}{a^{1/3}}\right)^2 \div \dfrac{a^{-1/3}}{b^{-1/2}}\right]^6 = \sqrt{b^3}$.

8. Show that $\sqrt[3]{(a + b)^5} \times (a + b)^{-2/3} = a + b$.

9. Show that $(x^{1/2} - 7)(x^{1/2} + 3) = x - 4\sqrt{x} - 21$.

10. Show that $\dfrac{x - 7x^{1/2}}{x - 5\sqrt{x} - 14} \div \left(1 + \dfrac{2}{\sqrt{x}}\right)^{-1} = 1$.

Chapter 7

FRACTIONAL EXPONENTS AND RADICALS

39. Fractional Exponents. Since $\sqrt{25} = 5$ and $5^2 = 25$, then $(\sqrt{25})^2 = 25$. Similarly, $(\sqrt{a})^2 = a$, $(\sqrt[3]{8})^3 = 8$, $(\sqrt[3]{a})^3 = a$, etc. That is, any root of any number (algebraic or other) raised to a power whose exponent is the same as the index of the root, gives the number itself. Thus,

$$(\sqrt{a})^2 = a, \quad (\sqrt[3]{a})^3 = a, \quad (\sqrt[4]{a})^4 = a, \text{ etc.}$$

But by relation (3) in article 35 we have also

$$(a^{1/2})^2 = a, \quad (a^{1/3})^3 = a, \quad (a^{1/4})^4 = a, \text{ etc.}$$

Therefore, $a^{1/2}$ is the same as $\sqrt{a}$, $a^{1/3}$ is the same as $\sqrt[3]{a}$, $a^{1/4} = \sqrt[4]{a}$, etc., and, in general,

$$a^{1/m} = \sqrt[m]{a}. \tag{8}$$

That is, a *fractional exponent* with 1 as numerator *means the same as a root* whose index is the denominator of the fractional exponent. For example, $(125)^{1/3} = \sqrt[3]{125} = 5$, $(3ab^2)^{1/7} = \sqrt[7]{3ab^2}$, etc.

Consider next the operation indicated by $\sqrt[3]{8^2}$. Since $8^2 = 64$ this means $\sqrt[3]{64}$, which is 4. Hence, $\sqrt[3]{8^2} = 4$. Consider also the operation indicated by $(\sqrt[3]{8})^2$. Since $\sqrt[3]{8} = 2$, this means 2^2, which is also 4. That is, $(\sqrt[3]{8})^2 = 4$. Since both these expressions are equal to 4 they are equal to each other. That is,

$$(\sqrt[3]{8})^2 = \sqrt[3]{8^2}.$$

In the same way we find that

$$(\sqrt[2]{9})^4 = \sqrt[2]{9^4} = 81,$$

and if a is any number whatever, and m and n are exponents and root indexes, we have, in general,

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}. \tag{9}$$

But, according to formula (8), $\sqrt[n]{a} = a^{1/n}$, and according to (3),

$(a^{1/n})^m = a^{1/n \times m} = a^{m/n}$. Therefore, $(\sqrt[n]{a})^m = a^{m/n}$. Using this form of the expression on the left side of (9) we have

$$a^{m/n} = \sqrt[n]{a^m}. \tag{10}$$

This states that any fractional exponent is equivalent to the root indicated by the denominator, of the power indicated by the numerator. That is, in a fractional exponent the numerator represents a power and the denominator a root, both applied to the same number or other expression to which the fractional exponent applies. Thus, according to (10), since a represents *any* real number, we can write $a^{2/3} = \sqrt[3]{a^2}$, $\sqrt[3]{b^6} = b^{6/3} = b^2$, $\sqrt{16a^2b^4} = 4ab^2$, and by (8) and (9) together $\sqrt[4]{16x^2y^8} = \sqrt[4]{16}x^{2/4}y^{8/4} = 2x^{1/2}y^2$.

If, in relations (4) and (5), article 35, the exponent m is replaced by a fractional exponent, say $1/n$, they become

$$(ab)^{1/n} = a^{1/n}b^{1/n}, \quad \left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}}.$$

But these fractional exponents are the same as roots, according to (8). Therefore,

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b} \tag{11}$$

and

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}. \tag{12}$$

These state that a root of a product or quotient equals, respectively, the product or quotient of the roots of the factors.

Formula (12) in particular states that in order to find a root of a fraction the roots of numerator and denominator are to be taken separately.

Fractional exponents may be either positive or negative and they obey the same laws as do positive and negative whole number exponents. Thus, according to formula (1), article 35,

$$\left.\begin{aligned} & & a^{1/m} \times a^{1/n} &= a^{1/m+1/n}, \\ &\text{by (2)} & \frac{a^{1/m}}{a^{1/n}} &= a^{1/m-1/n}, \\ &\text{by (3)} & (a^{1/m})^{1/n} &= a^{1/mn}, \\ &\text{and by (7)} & a^{-1/n} &= \frac{1}{a^{1/n}}. \end{aligned}\right\} \tag{13}$$

Examples of these relations are: $a^{1/3} \times a^{1/2} = a^{1/3+1/2} = a^{5/6}$, which according to (8) is the same as $\sqrt[6]{a^5}$; $\frac{4^{1/2}}{4^{1/3}} = 4^{1/2-1/3} = 4^{1/6} = \sqrt[6]{4}$; $(2b^{1/3})^{1/4} = 2^{1/4}b^{1/3\times 1/4} = 2^{1/4}b^{1/12} = \sqrt[4]{2} \times \sqrt[12]{b}$; $9^{-1/2} = \frac{1}{9^{1/2}} = \frac{1}{\sqrt{9}}$ $= \frac{1}{3}$; etc.

40. Multiplication and Division of Radicals. In algebra and its applications, we are frequently called on to find the value of such expressions as $\sqrt{3} \times \sqrt{12}$, $\sqrt[3]{16} \times \sqrt[3]{4}$, $\sqrt{x+y} \times \sqrt{2ab^3}$, $\frac{\sqrt{63}}{\sqrt{7}}$, etc. Since these roots can be expressed as exponents, and since we already have the laws governing operations with exponents, these products are easily found.

From relation (11) in the preceding article, $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$. Therefore, in the examples referred to above $\sqrt{3} \times \sqrt{12} = \sqrt{36} = 6$. Here neither of the separate square roots can be extracted exactly but by properly combining the radicals the product of $\sqrt{3}$ by $\sqrt{12}$ is readily found to be exactly 6. Similarly, $\sqrt[3]{16} \times \sqrt[3]{4} = \sqrt[3]{64} = 4$.

As a check on the results of this method, let us find the product of $\sqrt{4}$ by $\sqrt{25}$. According to the principle just used this is equal to $\sqrt{4 \times 25} = \sqrt{100} = 10$. But, separately, $\sqrt{4} = 2$ and $\sqrt{25} = 5$. Therefore, $\sqrt{4} \times \sqrt{25} = 2 \times 5 = 10$, as before.

Similarly, $\sqrt{x+y} \times \sqrt{2ab^3} = \sqrt{2ab^3(x+y)}$, $\sqrt[4]{35x^2y} \times \sqrt[4]{2a^3b^2}$ $= \sqrt[4]{70a^3b^2x^2y}$, etc.

This principle does not apply to the products of roots with different indexes. Thus, we cannot say that $\sqrt[3]{a} \times \sqrt[4]{b}$ is equal to either the third or fourth root of ab.

The relation (12) is also useful in handling radicals. It states that when n is any root index, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$. Thus, the expression given above, $\frac{\sqrt{63}}{\sqrt{7}}$, becomes $\sqrt{\frac{63}{7}} = \sqrt{9} = 3$. Here neither $\sqrt{63}$ nor $\sqrt{7}$ can be found exactly but by using the formula the value of the given expression is found exactly and immediately. In the same way $\frac{\sqrt[3]{280}}{\sqrt[3]{35}} = \sqrt[3]{\frac{280}{35}} = \sqrt[3]{8} = 2$, etc.

41. Reduction of Radicals. The square root of the number 75 cannot be found exactly, that is, there is no number which when multiplied by itself gives 75. Approximately, however, $\sqrt{75} = 8.66$. Even though $\sqrt{75}$ cannot be expressed exactly, however, it can, so to speak, be reduced to "lower terms," if we may use the words usually applied to reduction of fractions. This is accomplished by the use of the principle (11). Since $75 = 25 \times 3$ we can write $\sqrt{75} = \sqrt{25 \times 3}$ and by (11) $\sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3}$. But $\sqrt{25} = 5$: therefore,

$$\sqrt{75} = 5\sqrt{3},$$

which means "5 times $\sqrt{3}$."

In the same manner $\sqrt[3]{135}$, which cannot be found exactly but is approximately equal to 5.13, can be written as $\sqrt[3]{27 \times 5}$, and by (11) this is equal to $\sqrt[3]{27} \times \sqrt[3]{5}$ which is $3\sqrt[3]{5}$. Hence,

$$\sqrt[3]{135} = 3\sqrt[3]{5}.$$

Similarly, $\sqrt{1125} = \sqrt{225 \times 5} = \sqrt{225} \times \sqrt{5} = 15\sqrt{5}$; $\sqrt[5]{972} = \sqrt[5]{243 \times 4} = \sqrt[5]{243} \times \sqrt[5]{4} = 3\sqrt[5]{4}$; etc.

Even when a root of a number can be found exactly, it can, in some cases, be found with less trouble by the method just given than by the usual rules for such roots as the square and cube roots. As an example, let us try to find $\sqrt{1936}$. By trial this number is found to be the product of 16 and 121. Therefore,

$$\sqrt{1936} = \sqrt{16 \times 121} = \sqrt{16} \times \sqrt{121} = 4 \times 11 = 44.$$

Similarly,

$$\sqrt{8281} = \sqrt{49 \times 169} = \sqrt{49} \times \sqrt{169} = 7 \times 13 = 91;$$

$$\sqrt[3]{216} = \sqrt[3]{8 \times 27} = \sqrt[3]{8} \times \sqrt[3]{27} = 2 \times 3 = 6;$$

$$\sqrt[3]{13{,}824} = \sqrt[3]{64 \times 216} = \sqrt[3]{64} \times \sqrt[3]{216} = 4 \times 6 = 24, \text{ etc.}$$

Of course, this method of root extraction only applies when the number whose root is to be found can be expressed as the product of factors whose corresponding roots can be found exactly. In this connection, it is to be noted that formula (11) holds good not only for two factors but for any number of factors. Thus,

$$\sqrt[n]{abc} = \sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c}, \text{ etc.}$$

The third of the relations (13) enables us to find higher roots for which there are no definite rules. Thus, when the two arithmetic rules for the square and cube roots are known, we can find the fourth, sixth, ninth, etc., roots, and any root whose index is a power of 2 or 3 or contains only 2 and 3 as factors. The method is based on the following considerations.

The formula is

$$a^{1/mn} = (a^{1/m})^{1/n}.$$

But, $a^{1/mn} = \sqrt{a}$ and $a^{1/m} = \sqrt[m]{a}$; therefore,

$$\sqrt[mn]{a} = (\sqrt[m]{a})^{1/n} = \sqrt[n]{(\sqrt[m]{a})}.$$

This means that if a, m and n are any real numbers, the mnth root of a is the nth root of the mth root of a. Thus, $4 = 2 \times 2$ and the fourth root is the square root of the square root; $6 = 2 \times 3$ and the sixth root is the square root of the cube root, etc. These are written

$$\left.\begin{aligned} &\sqrt[4]{a} = \sqrt{\sqrt{a}}, \quad \sqrt[6]{a} = \sqrt[9]{\sqrt[3]{a}}, \\ \text{Similarly,}\quad &\sqrt[8]{a} = \sqrt{\sqrt[4]{a}}, \quad \sqrt{a} = \sqrt[3]{\sqrt[3]{a}}, \text{ etc.} \end{aligned}\right\} \qquad (14)$$

For example,

$$\sqrt[4]{625} = \sqrt{\sqrt{625}} = \sqrt{25} = 5;$$

$$\sqrt[6]{262144} = \sqrt{\sqrt[3]{262144}} = \sqrt{64} = 8, \text{ etc.}$$

42. Addition and Subtraction of Radicals. An operation which is sometimes necessary is the addition and subtraction of radicals. It is performed by means of an application of the method of the preceding article, the radicals being first reduced to their lowest terms and "like" radicals being then combined, algebraically.

For example, suppose the two radicals $\sqrt{75}$ and $\sqrt{48}$ are to be added. Of course, $\sqrt{75} = 8.66$ and $\sqrt{48} = 6.93$ and the sum of the two is approximately 15.59. We have seen, however, that $\sqrt{75} = 5\sqrt{3}$; by the same method $\sqrt{48} = 4\sqrt{3}$. Therefore,

$$\sqrt{75} + \sqrt{48} = 5\sqrt{3} + 4\sqrt{3} = 9\sqrt{3}.$$

Similarly,

$$\sqrt{75} - \sqrt{48} = 5\sqrt{3} - 4\sqrt{3} = \sqrt{3}.$$

In this way, the algebraic sum of two or more radicals may be expressed

in terms of a single radical, if each radical in the sum can be reduced so that all are expressed in terms of the same radical.

As another example, let us try to simplify the following expression:

$$5\sqrt[3]{54} + \sqrt[3]{250} - 3\sqrt[3]{16}.$$

The several radicals can be written as follows:

$$\sqrt[3]{54} = \sqrt[3]{27 \times 2} = 3\sqrt[3]{2},$$

$$\sqrt[3]{250} = \sqrt[3]{125 \times 2} = 5\sqrt[3]{2},$$

$$\sqrt[3]{16} = \sqrt[3]{8 \times 2} = 2\sqrt[3]{2}.$$

Therefore, the given expression can be written as

$$5(3\sqrt[3]{2}) + 5\sqrt[3]{2} - 3(2\sqrt[3]{2})$$

or

$$15\sqrt[3]{2} + 5\sqrt[3]{2} - 6\sqrt[3]{2},$$

and we have, finally, when these are combined with 15, 5 and -6 as coefficients:

$$5\sqrt[3]{54} + \sqrt[3]{250} - 3\sqrt[3]{16} = 14\sqrt[3]{2}.$$

As a final example of this method,

$$\begin{aligned}\sqrt{2} \times \sqrt{62.5} + 3\sqrt{80} - 8\sqrt{20} + \sqrt{288} - 2\sqrt{3} \times \sqrt{24} + \sqrt{2}\\ &= \sqrt{125} + 3\sqrt{80} - 8\sqrt{20} + \sqrt{288} - 2\sqrt{72} + \sqrt{2}\\ &= 5\sqrt{5} + 3(4\sqrt{5}) - 8(2\sqrt{5}) + 12\sqrt{2} - 2(6\sqrt{2}) + \sqrt{2}\\ &= (5\sqrt{5} + 12\sqrt{5} - 16\sqrt{5}) + (12\sqrt{2} - 12\sqrt{2} + \sqrt{2})\\ &= \sqrt{5} + \sqrt{2}.\end{aligned}$$

The operations of this article and article 40 may be combined and polynomial factors or polynomial numerators, denominators, etc., reduced in multiplication, division, etc., of complicated expressions by reducing the individual radicals as in the preceding article and then performing the various combinations, thus allowing the simplification of the complicated expression.

A radical expression which has no exact root and which cannot be reduced to a radical indicating a root of a smaller number is called a *surd*. Thus $\sqrt[3]{17}$, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{3}$, $\sqrt[4]{11}$ are surds. When a number stands under a radical sign which indicates that a certain root is to be

taken, but the root is not actually extracted, the root is said to be an *indicated root.* Thus, $\sqrt{25}$ is the indicated square root of 25, the actual square root being 5. Similarly, $\sqrt[3]{75}$ is the indicated cube root of 75; its actual value is approximately 4.217. Thus, we may say that: A *surd* is an indicated root of a number which has no such exact root, and has no factor which has such an exact root, as the one indicated; or *a surd is an irreducible indicated root.*

The last several articles thus deal with surds. In some parts of advanced algebra the subject of surds is important, but it will not be considered further here.

43. Exercises.

Express the following with radical signs and positive integral exponents:

1. $x^{3/5}$; $a^{-1/2}$; $5x^{-1/2}$; $\dfrac{2}{b^{-1/4}}$; $\dfrac{c^{-1/3}}{2}$; $2a^{-1/x}$.

2. $a^{-1/3} \times 2a^{-1/2}$; $x^{-2/3} \div 2a^{-1/2}$; $7a^{-1/2} \times 3a^{-1}$; $\dfrac{2a^{-2}}{a^{-3/2}}$.

Express the following without radicals:

3. $\sqrt[3]{a^2} \times \sqrt{a^3}$; $\sqrt[a]{x} \div \sqrt[3a]{x^3}$; $\sqrt[4]{a^n} \times \sqrt[3]{a^n} \div \sqrt[12]{a^{5n}}$.

Find the value of each of the following:

4. $16^{3/4}$; $4^{-5/2}$; $125^{2/3}$; $8^{-2/3}$; $36^{-3/2}$; $243^{2/5}$.

5. Reduce to simplest form: $\sqrt{288}$; $\sqrt{147}$; $\sqrt[3]{250}$; $\sqrt[3]{432}$; $2\sqrt{720}$; $\sqrt{36a^3}$.

Perform the indicated operations and simplify:

6. $2\sqrt{14} \times \sqrt{21}$; $\sqrt{8} \times 3\sqrt{6}$; $\sqrt[3]{168} \times \sqrt[3]{147}$; $8\sqrt{12} \times 3\sqrt{24}$.
7. $\sqrt{10} \div \sqrt{2}$; $21\sqrt{392} \div 7\sqrt{98}$; $6\sqrt{\frac{21}{3}}\sqrt{14}$; $2\sqrt{\frac{432}{6}}\sqrt{3}$.

Find the value of:

8. $3\sqrt{45} - \sqrt{20} + 7\sqrt{5}$.
9. $4\sqrt{63} + 5\sqrt{7} - 8\sqrt{28}$.
10. $2\sqrt{363} - 5\sqrt{243} + \sqrt{192}$.

Chapter 8

IMAGINARY AND COMPLEX NUMBERS

44. Indicated Roots. We have seen in the last article that if a certain root of a number cannot be extracted exactly, as, for example, $\sqrt{2}$, which is approximately equal to 1.414, the number may be left under the radical sign and the root then referred to as an *indicated root.* An indicated root is in all respects, and may be used in all calculations, the same as the actual value of the root. Thus, we write

$$\sqrt{2} = 1.4142\ldots$$

and the symbol or number $\sqrt{2}$ means the same as, and is more easily and compactly written than, the cumbersome symbol 1.4142 . . . . We have also seen that such numbers may be multiplied, divided, added and subtracted. Thus, $\sqrt{2}$ and $\sqrt{4\frac{1}{2}}$ are indicated roots whose values we do not know exactly but in accordance with the results of article 40 we can at once write out their product. It is $(\sqrt{2}) \times (\sqrt{4\frac{1}{2}}) = 3$. Similarly, as in article 42, $(\sqrt{18}) + (\sqrt{32}) = 7\sqrt{2}$, and so on.

Although some indicated roots cannot be found exactly, they may still be found as closely as may be necessary. There are other indicated roots, however, which cannot be written in decimal form, even approximately. These must, therefore, always be expressed as indicated roots and are considered as a new kind of number. Thus, we have seen in article 31 that the square root of a negative number is not a real number. Similarly, no *even* root of a negative number is a real number. The indicated even roots of negative numbers, therefore, belong to a new class of numbers. These numbers are called *imaginary numbers.*

45. Definitions. An *imaginary number* is an indicated even root of a negative number. By the method of article 41 the fourth, sixth, eighth, and, similarly, any *even* root may be reduced to a square root.

The indicated *square root* of a negative number may, therefore, be taken as representing any and all imaginary numbers.

By the method of article 40, the square root of *any* number may be expressed as the product of two square roots. Thus,

$$\sqrt{-25} = \sqrt{25 \times (-1)} = \sqrt{25} \times \sqrt{-1} = 5\sqrt{-1}.$$

Similarly,

$$\sqrt{-121} = \sqrt{121 \times (-1)} = \sqrt{121} \times \sqrt{-1} = 11\sqrt{-1};$$

$$\sqrt{-18} = \sqrt{9 \times 2 \times (-1)} = \sqrt{9} \times \sqrt{2} \times \sqrt{(-1)}$$
$$= 3\sqrt{2} \times \sqrt{-1},$$

or,

$$\sqrt{-18} = 3 \times 1.414 \times \sqrt{-1} = 4.242\sqrt{-1};$$

$$\sqrt{-10} = \sqrt{10} \times \sqrt{-1} = 3.162\sqrt{-1}, \text{ etc.,}$$

and in general

$$\sqrt{-a} = \sqrt{a} \times \sqrt{-1}.$$

By this method, therefore, any imaginary number may be expressed as a multiple of the imaginary number $\sqrt{-1}$. The number $\sqrt{-1}$, is, therefore, called the *imaginary unit*. Since it is much used in higher pure and applied mathematics a single symbol is chosen to represent it, according to the usual method of algebra. This symbol is the letter i, the initial letter of the word "imaginary." (In electrical engineering where i represents electric current, $\sqrt{-1}$ is generally represented by the letter j, but in pure mathematics the custom is to use i.) This symbol is subject to all the usual rules of algebra and we shall use it hereafter. The definition of the imaginary unit i is, therefore.

$$i = \sqrt{-1}. \qquad (15)$$

To distinguish them from imaginary numbers all other numbers are called *real numbers*. (This distinction is unfortunate as the so-called "imaginaries" are in every sense except one just as real as the so-called "reals," but the names are relics of the times when mathematicians discarded and ignored all even roots of negatives. They are now, however, as useful as any other numbers.) Real numbers include the ordinary integers (whole numbers), fractions, all roots of positive numbers whether exact, approximate or indicated, and odd roots of

negatives. Both real and imaginary numbers may be positive or negative.

A number which is the algebraic sum of a real number and an imaginary number is called a *complex number*. Thus, $4 + \sqrt{-3}$, or $4 + 1.732\sqrt{-1}$, or $4 + 1.732i$, as it may be written, is a complex number. So are $5 - 3i$, $-\sqrt{7} + \sqrt{2}i$, $2x - 3yi$, $a + ib$, etc. The usual algebraic symbol for *any* complex number is

$$a + ib, \quad \text{or} \quad x + iy,$$

where a, b and x; y are positive or negative real numbers and $i = \sqrt{-1}$. Thus a complex number is, in general, a binomial. All the usual rules for addition, subtraction, multiplication, division, factorization, involution and evolution apply to complex binomials as to any other binomial, due allowance being made for the properties of i, as we shall see presently.

46. Properties of the Imaginary Unit. Since by definition the imaginary unit is $i = \sqrt{-1}$, then $i^2 = (\sqrt{-1})^2 = -1$. Then $i^3 = i^2 \times i = (-1) \times i = -i$, and $i^4 = (i^2)^2 = (-1)^2 = +1$. If we continue raising i to successive higher powers, we find in the same way $i^5 = i^4 \times i = (+1) \times i = +i$, $i^6 = -1$, $i^7 = -i$, $i^8 = +1$, etc. Thus, the values of powers of i repeat themselves in groups of four. That is, every fourth successive power of i is the same. This appears more clearly if we tabulate these results, as follows:

$$\left.\begin{array}{ll} i = +i & i^5 = +i \\ i^2 = -1 & i^6 = -1 \\ i^3 = -i & i^7 = -i \\ i^4 = +1 & i^8 = +1, \text{ etc.} \end{array}\right\} \qquad (16)$$

From this table of powers of i it is seen that all *even powers of i are* ± 1 and all *odd powers of i are* $\pm i$. Therefore, if in any algebraic operation any number is multiplied or divided by an even power of i the symbol i with its exponent may be omitted and the number preceded simply by a plus or a minus sign, and if multiplied or divided by an odd power of i the exponent of i may be omitted and the proper sign placed before it. Similarly, if powers of i appear alone as terms of a polynomial they may be replaced by 1 or i with the proper sign.

Consider next the fraction or number $\frac{1}{i}$. Now, in algebra, as in

arithmetic, the multiplication of *both* numerator and denominator of a fraction by the *same* number does not change the value of the fraction. We can, therefore, write $\frac{1}{i} = \frac{1 \times i}{i \times i} = \frac{i}{i^2}$. But, $i^2 = -1$ and hence $\frac{i}{i^2} = \frac{i}{-1} = -i$. That is,

and also
$$\left.\begin{aligned} \frac{1}{i} &= -i, \\ i &= -\frac{1}{i}. \end{aligned}\right\} \tag{17}$$

These relations tell us that in any expression i may be shifted from denominator to numerator, and vice versa, simply by changing the sign of the expression, or, division by i is the same as multiplication by $-i$ and vice versa.

47. Addition and Subtraction of Complex Numbers. Let $a + ib$ and $c + id$ represent two complex numbers. The sum of the two is then

$$(a + ib) + (c + id) = a + c + ib + id,$$

or, taking out the common factor i in the last two terms,

Similarly,
$$\left.\begin{aligned} (a + ib) + (c + id) &= (a + c) + i(b + d). \\ (a + ib) - (c + id) &= (a - c) + i(b - d). \end{aligned}\right\} \tag{18}$$

The sum (or difference) of two complex numbers is, therefore, another complex number whose real part is the sum (or difference) of the real parts of the two numbers and whose imaginary part is the sum (or difference) of their imaginary parts.

Since a complex number is composed of two parts which are always distinct, it cannot be zero unless *both parts are separately equal to zero.* This is an important property of complex numbers, which we now make use of to show another important property of these numbers.

Suppose two complex numbers are equal; then the difference between them is zero. That is, if $a + ib$ and $c + id$ are two complex numbers, and if

$$a + ib = c + id,$$

then

$$(a + ib) - (c + id) = 0.$$

But, according to (18) this difference is $(a - c) + i(b - d)$. Therefore,

$$(a - c) + i(b - d) = 0.$$

This is a complex number whose real part is $a - c$ and whose imaginary part is $b - d$, and it is equal to zero. But for this complex number to be equal to zero its real and imaginary parts are *each* equal to zero, as seen above. Therefore,

$$a - c = 0, \quad \text{and} \quad b - d = 0.$$

But, when the difference between two numbers is zero, the two are equal. Therefore,

$$a = c, \quad \text{and} \quad b = d.$$

Now a, c and b, d are the real and imaginary parts of the two original *equal* complex numbers, and they are, *respectively*, equal. We, therefore, have the important result that

If two complex numbers are equal their real parts must be equal and their imaginary parts equal.

This result may be expressed in symbols by saying that if

$$\left.\begin{aligned} &A = a + ib, \qquad B = c + id, \\ \text{then if} \quad &A = B, \quad a = c \quad \text{and} \quad b = d. \end{aligned}\right\} \tag{19}$$

For example, $4 + 3i$ is not equal to $3 + 4i$, nor is $4 + 3i$ the same as $4 - 3i$.

From this principle it is seen that we cannot speak in the usual sense of one complex number being greater (or less) than another.

We shall see presently that there is another sense in which we can refer to the so-called "absolute" value of a complex number and compare these absolute values in the ordinary sense.

If we use the same form of expression for the results (18) as is used above for (19) we can say that: If

$$\left.\begin{aligned} &A = a + ib, \quad B = c + id, \\ &A + B = x + iy, \\ \text{where} \quad &x = a + c, \qquad y = b + d. \end{aligned}\right\} \tag{19a}$$

This method of expressing complex numbers and relations between them is frequently useful.

48. Multiplication and Division of Complex Numbers. The product of two complex numbers $a + ib$ and $c + id$ is found in the same way as that of any two binomials. Thus:

$$\begin{array}{r} a + ib \\ \underline{c + id} \\ iad + i^2bd \\ \underline{ac + ibc \qquad\qquad\quad} \\ ac + i(ad + bc) + i^2bd \end{array}$$

But $i^2 = -1$ and, therefore, $i^2bd = -ba$. The product is, therefore, $ac + i(ad + bc) - bd$, or

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

This is also a complex number whose real part is $ac - bd$ and whose imaginary part is $ad + bc$. The product of any two complex numbers, is, therefore, another complex number whose real part is the *difference between the product of the real and that of the imaginary parts* of the factors, and whose imaginary part is the *sum of the cross products of the real and imaginary parts* of the factors.

This result is expressed in symbols by saying that, if

$$\left.\begin{array}{ll} & A = a + ib, \quad B = c + id, \\ & AB = x + iy, \\ \text{where} & x = ac - bd, \quad y = ad + bc. \end{array}\right\} \tag{20}$$

The square of the complex number $a + ib$ is written in the usual way and has the same meaning as the square of any binomial, that is,

$$(a + ib)^2 = (a + ib) \times (a + ib).$$

If we apply the multiplication rule (20) to this product we find that,

$$(a + ib)(a + ib) = (a^2 - b^2) + i(ab + ab),$$

or,

$$(a + ib)^2 = (a^2 - b^2) + i \cdot 2ab.$$

In the same way we find that,

$$(a - ib)^2 = (a^2 - b^2) - i \cdot 2ab.$$

Both these results may be concisely expressed by letting

$$\left.\begin{array}{ll} & A = a \pm ib; \\ \text{then} & A^2 = x \pm iy, \\ \text{where} & x = a^2 - b^2, \quad y = 2ab. \end{array}\right\} \tag{21}$$

The quotient of two complex numbers $A = a + ib$ and $B = c + id$ is written in the usual form

$$\frac{A}{B} = \frac{a + ib}{c + id}.$$

As seen before, this fraction may have its numerator and denominator multiplied by the same number without changing its value. Let us multiply by $c - id$. This gives

$$\frac{A}{B} = \frac{(a + ib)(c - id)}{(c + id)(c - id)}.$$

By the method given above, the product in the numerator of the fraction on the right is found to be $(ac + bd) + i(bc - ad)$ and according to the result (4), article 17, the product in the denominator (product of sum and difference of same numbers) is $c^2 - i^2d^2$, or, since $i^2 = -1$, $c^2 + d^2$. The quotient is, therefore,

$$\frac{A}{B} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2},$$

that is,

$$\frac{A}{B} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right).$$

The quotient of two complex numbers is, therefore, also a complex number.

The result just obtained can be expressed by writing

$$\left.\begin{array}{ll} & A = a + ib, \quad B = c + id, \\ \text{and} & A/B = x + iy, \\ \text{where} & x = \dfrac{ac + bd}{c^2 + d^2}, \quad y = \dfrac{bc - ad}{c^2 + d^2}. \end{array}\right\} \tag{22}$$

49. Absolute Value of a Complex Number. In finding an expression for the quotient of two complex numbers in the preceding article, we made use of a product of the form

$$(a + ib)(a - ib) = a^2 + b^2.$$

Now, this result contains no imaginary part. Therefore, we have here a real number as the product of two complex numbers. This special case does not conform to the general rule for multiplication of complex numbers and is of particular significance.

It is to be noted that the two complex numbers $a + ib$ and $a - ib$

are closely related in a very simple manner: they differ only in the sign of their imaginary parts. Two complex numbers related in this manner are said to be *conjugate* complex numbers. The conjugate of the complex number A is represented by $\overline{A}$ (called "A bar"). Thus, the conjugate of the complex number

$$\left.\begin{aligned} A &= a + ib \\ \overline{A} &= a - ib \end{aligned}\right\} \tag{23}$$

is (as above)

and their product is

$$A\overline{A} = a^2 + b^2.$$

The square root of this product is also a real number and is called the *absolute value* of the complex number A. This is written $|A|$. Using these symbols and the values of A and $\overline{A}$ as given by (23)

$$\left.\begin{aligned} |A|^2 = A\overline{A} &= a^2 + b^2, \\ \therefore \quad |A| &= \sqrt{a^2 + b^2}. \end{aligned}\right\} \tag{24}$$

Similarly, if $B = c + id$, the absolute value of B is $|B| = \sqrt{c^2 + d^2}$; if $A = 4 + 3i$, $|A| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$. While the absolute value, being a square root, may be written as either plus or minus, the positive value is always taken. Thus, when

$$A = 4 + 3i, \quad |A| = 5.$$

Two complex numbers may be said to be equal *in absolute value*, or one may be said to be greater or less than another in absolute value, in the usual sense of speaking of positive numbers.

Complex numbers are of the greatest importance in higher mathematics, where a large branch of the subject is called the "theory of functions of a complex variable," and in many branches of science and engineering.

50. Exercises.

Use the symbol i in expressing the results of the following:

1. Simplify: $\sqrt{-8} + \sqrt{-18}$; $4\sqrt{-27} + 3\sqrt{-12}$; $5\sqrt{-16} - 2\sqrt{9}$.

2. Multiply: $(\sqrt{-3})(\sqrt{-12})$; $(2 + \sqrt{-2})(1 - \sqrt{-3})$; $(2i\sqrt{2} + i\sqrt{3})$ by $(i\sqrt{3} - i\sqrt{5})$; $(2 + 5i)(5 - 2i)$.

3. Divide: $\sqrt{27}$ by $\sqrt{-3}$; $(-\sqrt{-4}) \div (\sqrt{-2} + \sqrt{-3})$; $\dfrac{\sqrt{-a^2} + \sqrt{-2}}{2\sqrt{-1} - \sqrt{-2}}$; $\dfrac{\sqrt{5} - 3i}{2 + \sqrt{3}i}$.

4. Express in the form $x + iy$: $\dfrac{3 + 5i}{2 + 3i}$; $\dfrac{1 + i}{1 - i}$; $\dfrac{2\sqrt{3} - i\sqrt{2}}{\sqrt{3} + i\sqrt{2}}$.

5. Express in the form $x + iy$: $(2 + 3i)(4 + 5i)$; $(3.5 - 4i)(2 + 1.25i)$; $(\sqrt{8} + i\sqrt{32})(\sqrt{2} + i\sqrt{1\frac{1}{8}})$.

6. Find the absolute value of the complex numbers $A = 6 + 8i$; $B = 12 - 5i$; $C = 3 + 5i$.

7. Find the absolute values of the answers to Exercise 4 above.

Perform the following indicated operations:

8. $(2 + 3i) \div [(2i - 1)(5i - 3)]$.

9. $(-2 + i)(3 - 2i)(-4 + 3i)$.

10. $\dfrac{(2 + i)(1 + 2i)(1 + 3i)}{(1 + i)(2 - 2i)(1 + 3i)}$.

Chapter 9

POWERS AND ROOTS OF BINOMIALS

51. Introduction. In article 17 we have found the complete expressions for several powers of a binomial such as $a + b$, where a and b may be any numbers whatever. Thus, we found that $(a + b)^2 = a^2 + 2ab + b^2$, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, etc. Also in Chapter 5 we found that there is no simple way of expressing the value of $\sqrt{a + b}$, $\sqrt[3]{a + b}$, $\sqrt[3]{a^2 + b^2}$, etc., in terms of powers or roots of the separate terms a, b, a^2, etc. According to the principles of fractional exponents, however, $\sqrt{a + b} = (a + b)^{1/2}$, $\sqrt[3]{a + b} = (a + b)^{1/3}$, $\sqrt{a^2 + b^2} = (a^2 + b^2)^{1/2}$, etc., that is, *these are powers of binomials.*

If, therefore, we could write out the expression for *any* power of a binomial we would then be able to write expressions for these indicated roots. Similarly, we could write $\dfrac{1}{(a + b)^2} = (a + b)^{-2}$, $\dfrac{1}{\sqrt[3]{a + b}} = \dfrac{1}{(a + b)^{1/3}} = (a + b)^{-1/3}$, etc., and then write out the complete expression for any one of these as a power of a binomial. Also, even though such a number as, say, 13 has no exact square root it can be written as a binomial, $7 + 6$, $10 + 3$, etc., and by the use of such a principle we could write out the value of the square root of 13 by writing it as

$$\sqrt{13} = \sqrt{10 + 3} = (10 + 3)^{1/2},$$

a power of a binomial.

From these considerations, it appears that any power or root of a binomial or quotients involving these can be expressed as a power of a binomial, and any rule or principle by means of which we could write out the expression for the power would serve for any of these expressions.

Since any number, algebraic or other, positive or negative, integral

or fractional, real or imaginary, may be represented by such symbols as a, b and any exponent, be it integral, fractional or mixed number, positive or negative, may be represented by n, let us say, then any power of any binomial may be represented by the expression $(a + b)^n$. If then we can obtain an expression for $(a + b)^n$, we can write out expressions for any powers, roots, quotients, etc., such as those cited above. The rule or principle by means of which this is done is called the *binomial theorem.* This we will now proceed to develop.

52. The Binomial Theorem. Let us write out here a few of the powers of the binomial $a + b$ as found in article 17. We have

$$(a + b)^2 = a^2 + 2ab + b^2,$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Also,

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5,$$

and so on, as found by continued multiplication by $(a + b)$. Let us notice these polynomials carefully.

It is noticed in each case that

1. The number of terms is 1 greater than the exponent of $(a + b)$.
2. The first term contains only the number a, and its exponent is the same as that of $(a + b)$. In the remaining terms a appears with exponents in descending order.
3. The number b appears first in the second term with exponent 1 and in the remaining terms with exponents in ascending order, the last term containing b alone with exponent the same as that of $(a + b)$.
4. The coefficient of the first and last terms is 1 and that of the second and next-to-last terms is equal to the exponent of $(a + b)$.
5. The coefficients of the terms seem to follow some definite rule of formation.

The rule for forming the coefficients is not at once obvious but it can easily be found. Thus, in the expression for $(a + b)^4$, let us examine the coefficients of the second and third terms. The coefficient of the second term is 4, the exponent of a in that term is 1 less, that is, 3, and that of b is 1. Suppose we multiply the coefficient by the exponent of a, this gives $4 \times 3 = 12$. Now, divide this product by the b exponent plus 1, that is, by 2. The result is 6, which is the

coefficient of the next term. Next, consider the third term. Here also multiply the coefficient by the a exponent, and divide the product by the b exponent plus 1. The result is $(6 \times 2) \div 3 = 4$, which is again the *coefficient of the next term.* It will be found by trial that this method will serve to form any coefficient after the first, from the preceding coefficient in any of the polynomials given above. The rule for the coefficients is, therefore, the following:

In any term multiply the coefficient by the a exponent and divide the product by the b exponent plus 1; *the result is the coefficient of the next term.*

As an example, find the next to last coefficient of the expression for $(a + b)^5$. Applying the rule to the preceding term we get $\dfrac{2 \times 10}{3 + 1} = \dfrac{20}{4} = 5$, which is correct as seen.

We now have the complete scheme for writing out all the terms for any power of $(a + b)$. Let us, therefore, put n for the exponent and write out the rule for the expression for $(a + b)^n$. Summarizing the above results we have:

1. *The number of terms in* $(a + b)^n$ *is* $n + 1$.
2. *The exponent of a in the first term is n and decreases by* 1 *in each succeeding term.*
3. *The number b first appears in the second term with exponent* 1 *and the exponent increases by* 1 *in each succeeding term.*
4. *The first term coefficient is* 1 *and the second is n.*
5. *The other coefficients are found as follows: In any term multiply the coefficient by the a exponent and divide by the b exponent plus* 1; *the result is the coefficient of the next term.*

Writing out the expression for $(a + b)^n$, therefore, the first term is a^n and the second is $n \cdot a^{n-1}b$. From this term the coefficient of the next term is found by the rule to be $n(n - 1) \div 2$, and the term is, therefore, $\dfrac{n(n - 1)}{2} \cdot a^{n-2}b^2$. From this term the coefficient of the next is found to be $\left\{\left[\dfrac{n(n - 1)}{2}\right](n - 2)\right\} \div 3 = \dfrac{n(n - 1)(n - 2)}{2 \times 3}$, and the term is, therefore, $\dfrac{n(n - 1)(n - 2)}{2 \times 3} \cdot a^{n-3}b^3$. The first four terms are, therefore,

$$a^n, \quad n \cdot a^{n-1}b, \quad \frac{n(n - 1)}{2} \cdot a^{n-2}b^2, \quad \frac{n(n - 1)(n - 2)}{2 \times 3} \cdot a^{n-3}b^3,$$

and from these it is seen that the expression runs as follows (with 2 written as 1×2, 2×3 as $1 \times 2 \times 3$, etc.):

$$(a+b)^n = a^n + n \cdot a^{n-1}b + \frac{n(n-1)}{1 \times 2} \cdot a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} \cdot a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4} \cdot a^{n-4}b^4 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \times 2 \times 3 \times 4 \times 5} \cdot a^{n-5}b^5 + \ldots,$$

the string of dots indicating that other terms follow those written.

This form is easily remembered by noting the manner in which the exponents of a and b, and the factors in the numerator and denominator of the coefficient, appear in each successive term.

The continued product of all the whole numbers from 1 up to and including any number is called the factorial product or sinply the *factorial* of that number. Thus, $1 \times 2 \times 3 \times 4 \times 5$ is "factorial 5"; this is written $\lfloor 5$ or $5!$. Similarly, $1 \times 2 \times 3 = \lfloor 3$ and for any number m, $1 \times 2 \times 3 \times \ldots \times m = \lfloor m$.

Using this notation, our expression for $(a+b)^n$ can be written somewhat more simply. Writing only a few terms it is

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\lfloor 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\lfloor 3} a^{n-3}b^3 + \ldots \quad (25)$$

When the numerical value of n is a specified whole number, that is, $n = 1, 2, 3, 4$, etc., the number of terms in the expression (25) is determined and is equal to $n+1$. Thus, $(a+b)^2$ has 3 terms; $(a+b)^3$ has 4 terms; $(a+b)^4$ has 5 terms, etc. When the numerical value of n is not stated, however, the succession of terms in (25) goes on indefinitely and is called a *series*. If the exponent n is fractional, that is, represents a root, and the binomial does not have that root exactly, then the series never stops. In this case, the series is called an *infinite series*. Even though the binomial may not have the exact

root indicated by the fractional exponent, the root may be found as closely as may be desired by writing out as many terms of the series as may be necessary.

When the definite polynomial or the indefinite series of terms in the expression for a power of a binomial is written out according to (25) the binomial is said to be *expanded.*

The series (25) is the expression in symbols of the rule given above in italics for finding any power of a binomial. As already stated this rule or principle is called the *Binomial Theorem.* It is one of the most famous of mathematical rules and was discovered nearly three hundred years ago by Sir Isaac Newton, the celebrated English mathematician, physicist and philosopher.

53. How to Use the Binomial Theorem. As illustrations of the method of using the binomial theorem we shall write out the expressions for various powers of a few binomials.

Let us first expand $(a + b)^6$. Here the exponent is $n = 6$ and when 6 is used instead of n in (25) we have:

$$(a+b)^6 = a^6 + 6a^{6-1}b + \frac{6(6-1)}{\lfloor 2} a^{6-2}b^2 + \frac{6(6-1)(6-2)}{\lfloor 3} a^{6-3}b^3$$

$$+ \frac{6(6-1)(6-2)(6-3)}{\lfloor 4} a^{6-4}b^4$$

$$+ \frac{6(6-1)(6-2)(6-3)(6-4)}{\lfloor 5} a^{6-5}b^5$$

$$+ \frac{6(6-1)(6-2)(6-3)(6-4)(6-5)}{\lfloor 6} a^{6-6}b^6$$

$$= a^6 + 6a^5b + \frac{6 \times 5}{1 \times 2} a^4b^2 + \frac{6 \times 5 \times 4}{1 \times 2 \times 3} a^3b^3$$

$$+ \frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} a^2b^4 + \frac{6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5} a^1b^5$$

$$+ \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6} a^0b^6.$$

$$\therefore \quad (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

To expand $(a - b)^4$ we have $n = 4$ and b is negative. Hence,

$$(a - b)^4 = a^4 + 4a^{4-1}(-b) + \frac{4(4-1)}{\lfloor 2} a^{4-2}(-b)^2$$

$$+ \frac{4(4-1)(4-2)}{\lfloor 3} a^{4-3}(-b)^3$$

$$+ \frac{4(4-1)(4-2)(4-3)}{\lfloor 4} a^{4-4}(-b)^4$$

$$= a^4 - 4a^3b + \frac{4 \times 3}{1 \times 2} a^2b^2 + \frac{4 \times 3 \times 2}{1 \times 2 \times 3} a^1(-b^3)$$

$$+ \frac{4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4} a^0b^4$$

$$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

To expand $(3x + 2y)^3$ we have $3x$ in the place of a and $2y$ in the place of b, and $n = 3$. Hence,

$$(3x + 2y)^3 = (3x)^3 + 3(3x)^{3-1}(2y) + \frac{3(3-1)}{\lfloor 2} (3x)^{3-2}(2y)^2$$

$$+ \frac{3(3-1)(3-2)}{\lfloor 3} (3x)^{3-3}(2y)^3$$

$$= (3x)^3 + 3(3x)^2(2y) + \frac{3 \times 2}{1 \times 2} (3x)^1(2y)^2$$

$$+ \frac{3 \times 2 \times 1}{1 \times 2 \times 3} (3x)^0(2y)^3$$

$$= (3x)^3 + 3(3x)^2(2y) + 3(3x)(2y)^2 + (2y)^3.$$

$$\therefore \quad (3x + 2y)^3 = 27x^3 + 54x^2y + 36xy^2 + 8y^3.$$

In order to find $\sqrt{2a + 3x}$ by means of the binomial theorem we write it $(2a + 3x)^{1/2}$ and have $2a$ and $3x$ as the terms of the binomial with $n = \frac{1}{2}$. This expression has no exact square root and so the series goes on indefinitely, but we may stop the expansion whenever we wish to do so. We will write out enough terms to see how they run. Substituting in (25) we have:

$$(2a + 3x)^{1/2} = (2a)^{1/2} + \tfrac{1}{2}(2a)^{1/2-1}(3x) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{\lfloor 2} (2a)^{1/2-2}(3x)^2$$

$$+ \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{\lfloor 3} (2a)^{1/2-3}(3x)^3 + \dots$$

$$= (2a)^{1/2} + \tfrac{1}{2}(2a)^{-1/2}(3x) + \frac{\frac{1}{2}(-\frac{1}{2})}{1 \times 2}(2a)^{-3/2}(3x)^2$$

$$+ \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1 \times 2 \times 3}(2a)^{-5/2}(3x)^3 + \dots$$

$$= (2a)^{1/2} + \frac{1}{2}\frac{3x}{(2a)^{1/2}} + \frac{(-\frac{1}{4})}{2}\frac{(3x)^2}{(2a)^{3/2}} + \frac{(\frac{3}{8})}{6}\frac{(3x)^3}{(2a)^{5/2}} + \dots$$

$$= (2a)^{1/2} + \frac{3x}{2(2a)^{1/2}} - \frac{9x^2}{8(2a)^{3/2}} + \frac{27x^3}{16(2a)^{5/2}} - \dots$$

$$\therefore \quad \sqrt{2a + 3x} = \sqrt{2a} + \frac{3x}{2\sqrt{2a}} - \frac{9x^2}{8\sqrt{(2a)^3}} + \frac{27x^3}{16\sqrt{(2a)^5}} - \dots$$

In this last form of the expansion we have changed all negative exponents to positive by the rules of article 37 and expressed all fractional exponents as radicals by the rules of article 39.

Let us find an expression for $\dfrac{1}{\sqrt{x+y}}$. This is equivalent to $\dfrac{1}{(x+y)^{1/2}} = (x+y)^{-1/2}$, in which the terms of the binomial are x and y and $n = -\frac{1}{2}$. The binomial expansion, therefore, gives

$$(x+y)^{-1/2} = x^{-1/2} + (-\tfrac{1}{2})x^{-1/2-1}y + \frac{(-\frac{1}{2})(-\frac{1}{2} - 1)}{\underline{|2}}x^{-1/2-2}y^2$$

$$+ \frac{(-\frac{1}{2})(-\frac{1}{2} - 1)(-\frac{1}{2} - 2)}{\underline{|3}}x^{-1/2-3}y^3 + \dots$$

$$= x^{-1/2} - \tfrac{1}{2}x^{-3/2}y + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1 \times 2}x^{-5/2}y^2$$

$$+ \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1 \times 2 \times 3}x^{-7/2}y^3 + \dots$$

$$= x^{-1/2} - \tfrac{1}{2}x^{-3/2}y + \frac{(\frac{3}{4})}{2}x^{-5/2}y^2 - \frac{(\frac{15}{8})}{6}x^{-7/2}y^3 + \dots$$

$$= \frac{1}{x^{1/2}} - \frac{1}{2}\frac{y}{x^{3/2}} + \frac{3}{8}\frac{y^2}{x^{5/2}} - \frac{5}{16}\frac{y^3}{x^{7/2}} + \dots$$

$$\therefore \quad \frac{1}{\sqrt{x+y}} = \frac{1}{\sqrt{x}} - \frac{y}{2\sqrt{x^3}} + \frac{3y^2}{8\sqrt{x^5}} - \frac{5y^3}{16\sqrt{x^7}} + \dots$$

From these illustrations it is now easily seen that in order to find the expansion, either exact or approximate, of any expression to

which the binomial theorem can be applied the procedure is somewhat as follows:

1. *Write the expression in the form of a power of a binomial, using the laws of exponents in cases of fractional expressions or radicals.*

2. *Note from the binomial form the terms of the binomial with their signs and the value of the exponent, and write out the binomial expansion* (25) *with these values in the places of a, b, and n.*

3. *If n is a positive integer, the series will terminate automatically with $n + 1$ terms; otherwise, it goes on indefinitely and as many terms as desired may be written.*

4. *Reduce and simplify the separate terms, remembering the rules of signs and the laws of exponents.*

54. Exercises.

Expand the following by the binomial theorem:

1. $(x + y)^5$.
2. $(a - b)^5$.
3. $(2a + 3)^6$.
4. $(3x - 2y)^5$.
5. $(c^2 + b^2)^4$.
6. $(\frac{1}{2}x^2 + y^3)^4$.
7. $\dfrac{1}{1 - x}$ to five terms.
8. $\dfrac{1}{(1 - x)^2}$ to five terms.
9. $(1 + x)^n$ to five terms.
10. $\sqrt{1 + x}$ to five terms.
11. $\dfrac{1}{\sqrt{1 + x}}$ to five terms.
12. $\sqrt[3]{a + b}$ to five terms.

Part 3

EQUATIONS

Chapter 10

EQUATIONS AND THEIR SOLUTION

55. Meaning of Equation and Solution. If two algebraic expressions have the same value they are said to be equal, and the statement of this equality in symbols, by the use of the equality sign, is called an *equation*. Thus, the statements

$$2a + a = 4a - a,$$

and

$$(x - y)^2 = x^2 - 2xy + y^2$$

are equations. These two statements or equations are always true no matter what the values of a, x and y may be, for $2a + a$ is $3a$ and $4a - a$ is also $3a$, and we know that when $x - y$ is multiplied by $x - y$ the product is always $x^2 - 2xy + y^2$. An equation like the two above, which is a true statement for *all* values that the symbols in it may have, is called an *identical* equation, or an identity, that is, the statement is identically true.

Consider the two binomials $4x + 1$ and $8x - 7$. It cannot, in general, be said that these two expressions are equal. If, however, the symbol x has the particular value 2, the first binomial has the value $8 + 1 = 9$ and the second is $16 - 7$ which is also 9. In this particular case, the two binomials are equal. This can be written,

$$4x + 1 = 8x - 7 \qquad (1)$$

when

$$x = 2.$$

No other value of x beside 2 will make $4x + 1$ the same as $8x - 7$. The equation (1) is, therefore, a true statement only *on the condition that* $x = 2$. Equation (1) is, therefore, said to be a *conditional* equation, or an *equation of condition*. The value $x = 2$, then, fulfills or satisfies the condition expressed by the equation. This is stated by saying that the value $x = 2$ *satisfies the equation*. When a value of x has been found which satisfies the equation, the equation is said to be *solved*, and this value of x is called the *solution* or the *root* of the equation.

The process of solving the equation is also called the solution of the equation.

When a solution of an equation has been found, it can be tested for correctness by substituting it in place of the symbol in the equation. If the equation is then a true statement, the solution is correct. Thus, $x = 2$ is the correct solution of the equation (1) because the substitution of the value 2 for x in the equation gives $4(2) + 1 = 8(2) - 7$, or $8 + 1 = 16 - 7$, that is, $9 = 9$, which is an identity. The solution is, therefore, correct and by this procedure it is said to be *verified* or checked.

From this discussion it is seen that when an equation is not identical but conditional the value or values of the letter or letters in it are not known until the equation is solved. Unless otherwise specified equations are generally understood to be conditional, and the letters or symbols in equations are referred to as the *unknown quantities* of the equations, or more simply, the *unknowns*.

The two parts of an equation which are connected by the sign of equality (=) are called the *members*, or *sides* of the equation. The one on the left-hand side of the equation is called the left or *first* member, and that on the right is the right or *second* member. Thus, in equation (1), $4x + 1$ is the first and $8x - 7$ the second member.

When an equation contains only one unknown quantity, only the one condition is necessary to determine the value of the unknown. Thus, equation (1) states the single necessary and sufficient condition for determining the value of the unknown x, and $x = 2$ is the only value which will satisfy the equation. Suppose, however, we have the equation

$$x + y = 9.$$

This equation contains two unknowns and may be satisfied by any number of values of x and y. Thus, if $x = 8$ and $y = 1$, $x + y$ is 9; $x = 4$ and $y = 5$ also satisfy the equation, as do $x = 2\frac{1}{2}$ and $y = 6\frac{1}{2}$, $x = 11$ and $y = -2$, and so on. This means that the one condition that the sum of x and y shall be 9 is not sufficient to determine the values of x and y; one equation with two unknowns cannot be definitely and uniquely solved. Such an equation is called an *indeterminate* equation.

Suppose, however, that beside the condition $x + y = 9$, the condition $x - y = 5$ is also to be satisfied, the two being required to be

satisfied *at the same time.* It is then found that there is only one value of x, namely $x = 7$, and one value of y, namely $y = 2$, which jointly satisfy both conditions. The two equations together, therefore, definitely determine both x and y. That is, when

$$\left.\begin{aligned} x + y &= 9 \\ \text{and} \qquad x - y &= 5, \end{aligned}\right\}$$

the solutions are $x = 7$, $y = 2$.

From these discussions it is seen that only one equation is necessary to determine the value of one unknown, while two equations which must be satisfied *at the same time* are necessary in order to determine the values of two unknowns. Such pairs of equations are called *simultaneous equations.* Two simultaneous equations are said to be solved when the values of both unknowns are found. Similarly, three simultaneous equations are necessary to solve for three unknowns, and so on. In general, in order to determine any number of unknowns, as many simultaneous equations, that is, conditions, are required as there are unknowns.

56. Properties of Equations. If any two quantities are equal, and each is increased or decreased by the same amount, the results are still equal. Of course, neither is the *same* as at first, but the two are still *equal.* Thus if

$$(4x + 1) = (8x - 7)$$

then also

$$(4x + 1) + 3 = (8x - 7) + 3$$

for if $x = 2$ the equation as first written is equivalent to $9 = 9$, and the second to $9 + 3 = 9 + 3$, or $12 = 12$, which is, of course, true. Similarly, if

$$3y - 2 = 7$$

then also

$$(3y - 2) - 5 = 7 - 5.$$

Thus, we see that the same number, and also the same symbol, may be added or subtracted *on both sides* of an equation without changing the equality. This is sometimes expressed by saying that

If equals are added to equals the sums are equal, and *if equals are subtracted from equals, the remainders are equal.*

If two quantities or algebraic expressions are equal and both are exactly doubled it is obvious that the two new quantities or expressions

so obtained are also equal to each other, and, similarly, both may be exactly halved with equal results. In the same way *both* members of an equation may be multiplied or divided by *any* number without destroying the equality. Thus, in the case of equation (1),

$$4x + 1 = 8x - 7,$$

then

$$2(4x + 1) = 2(8x - 7)$$

and

$$\frac{4x + 1}{3} = \frac{8x - 7}{3}.$$

In general, we may say that

If equals are multiplied or divided by equals, the results are equal. Division by zero is excluded.

If both members of an equation are multiplied or divided by the same number, the equation is said to be multiplied or divided by that number.

If two numbers or algebraic expressions are equal, they, of course, have the same square roots and their squares are also the same, and, similarly, for any other power or root. This means that *both members* of an equation may be raised to any power or have the same root extracted and the results obtained will be equal. Thus,

Equal powers and roots of equals are equal.

The several properties of equations discussed above are of the greatest importance and will be used constantly in the succeeding chapters. In particular, let us notice the first one a little more closely at this point.

Consider the equation

$$3x + 4 = 7x - 8$$

and let us subtract 4 from both sides. Then

$$3x + 4 - 4 = 7x - 8 - 4,$$

or

$$3x = (7x - 8) - 4.$$

This equation now differs from the original in that the 4 which was on the left side at first seems to have been transferred to the right side and written with its original plus sign changed to minus.

Again let us add 8 to both sides of the original equation

$$3x + 4 = 7x - 8.$$

We have

$$3x + 4 + 8 = 7x - 8 + 8,$$

or

$$(3x + 4) + 8 = 7x.$$

Here the 8 seems to have been transposed from the right side of the original equation and to have had its sign changed from minus to plus.

Thus, it appears that the process of adding or subtracting a certain term on both sides of an equation at the same time is the same as simply shifting the equivalent of that term from the side it was originally on to the other side, and changing its sign. This is indeed the case and the process is called *transposition* of terms. We have, therefore, the important result that

Any term in either member of an equation may be transposed to the opposite member provided its sign is changed.

In the equation

$$4y + 7 = 14 - 3y$$

let us transpose the term $-3y$. The result is

$$4y + 7 + 3y = 14.$$

In this equation let us now transpose the $+7$,

$$4y + 3y = 14 - 7,$$

or

$$7y = 7.$$

Both these transpositions might have been made at once. In fact, if care is used the transposition and subsequent combination of terms may often be performed mentally and by inspection, and the final result written down at once. Thus, in the equation

$$4y - 3 = 2x + 2,$$

if we transpose the terms $2x$ and -3 and combine we get

$$4y - 2x = 5.$$

The operation of transposition will be used regularly in the succeeding chapters.

57. Types of Equations. In the equation

$$4x + 1 = 8x - 7$$

the exponent of the unknown x is 1 in both terms in which it appears. Such an equation is called a *simple equation*, or an equation of the *first*

degree. The word "simple" is not used here in the sense of "easy" but to indicate that the exponent of the unknown is "single," that is, equals 1. Similarly, the equation $3x - \frac{1}{2}y + z = 17\frac{3}{4}y - 2x + 15\frac{1}{2}$ also contains no power of either of the unknowns higher than 1 and is a simple equation.

In the equation

$$4x + 3x^2 - 2 = 7 + 3x,$$

however, the unknown x appears in two terms with the exponent 1 and in one term with the exponent 2. The exponent 2 is, however, the highest that does appear. On this account the equation is said to be of the *second degree.*

Now, the second power of any number is called its *square* and the square is a *four*-sided figure whose area is the algebraic square of the number representing the length of its side. The second power is, therefore, sometimes called the *quadratic* power, from the Latin prefix quad- for "four." An equation in which the highest power of the unknown is 2 is, therefore, called a *quadratic equation.* It is to be noted that the equation may contain both the first and second powers, as well as constant terms, or it may contain the second power alone. It is named from the *highest* power that occurs in the equation. Thus, $4x^2 - 4 = 12$ and $3y^2 + y = 4y + 3$ are both quadratic equations.

If the highest power of the unknown which appears in an equation is 3, that is, the cube of the unknown, the equation is of the *third degree* and is called a *cubic equation.* Thus,

$$x^3 - 7x = 4x^2 - 3x^3 + 2$$

and

$$x^3 + x = 1$$

are cubic equations.

An equation in which the highest power of the unknown is 4 is of the *fourth degree* and is called a *quartic* or *bi-quadratic* equation.

If the highest power of the unknown in an equation is 5 the equation is of the *fifth degree* and is called a *quintic equation.*

If the highest power of the unknown in an equation is 6 the equation is of the *sixth degree* and is sometimes called a *sextic* or *bi-cubic* equation.

Equations of higher degree than 6 do not have special names. Those of higher degree than 5 are usually referred to simply by stating the degree of the equation. In this book equations of the first, second, third, fourth and fifth degrees will be referred to respectively,

as simple, quadratic, cubic, quartic and quintic equations. Those of degree 6, 7, etc., will be referred to by stating the degree.

Equations of any degree, as well as the first, may be simultaneous, as many equations as unknowns being required in any case to determine the value of all the unknowns.

An equation of any degree is said to be *solved* when a value of the unknown is found which satisfies the equation. It will be seen later that equations of higher degree than the first with only one unknown have in general more than one solution.

In addition to equations which are designated by the degree of the term containing the highest power of the unknown, there is another type of equation in which the exponent itself is the unknown quantity. This type of equation is called an *exponential equation.* The equation $4^x - 60 = 4$ is an exponential equation with the exponent x as unknown.

58. Historical Sketch. The early history of algebra might almost be called the history of equations. In the earliest stages of its development algebra was hardly distinguished from arithmetic and was almost wholly devoted to the solution of problems by means of equations, although a few of the strictly algebraic principles which we have treated in Part II were recognized. Algebra was not properly distinguished from arithmetic until late in its history, the distinction being made only after the introduction of symbols to represent numbers. These did not at first occur in the forms which we have studied in Part I, however, until about the fourth century A.D. Even after this, arithmetic and algebra continued to develop together and the study of algebra consisted largely of the study of equations, a famous book which did much to advance the science of algebra bearing the title "Arithmetic" and the name *algebra* itself being derived from the (Arabian) word "al-gebr" which means the "addition and subtraction of the same quantity on both sides of an *equation.*"

The method of solution of simple equations seems to have been known from the earliest times, although they were stated in words instead of symbols and negative numbers were not recognized.

One of the earliest solutions of a quadratic equation of which there is a definite record occurs in the arithmetical section of the famous "Elements of Geometry" by Euclid, the mathematician who did more than any other one person to develop and fix the science of geometry. Euclid was a Greek who lived about 330–275 B.C. and flourished as a

teacher of mathematics at Alexandria in Egypt. The "Elements" seems to have been published in the later years of his life.

Euclid's solution of quadratic equations were accomplished without the use of symbols, lengths of lines being used to represent numbers, but expressed in the present-day notation the equations he solved were of the special forms $x^2 + ax = a^2$ and $x^2 + b = \pm ax$, where x is the unknown and a, b represent known constants. He did not discuss the general case in which the coefficients may be any numbers whatever.

The great work of antiquity on the subject of algebra was the "Arithmetic" by Diophantus of Alexandria, who seems to have lived during the first part of the fourth century A.D. and died at the age of 84. In his book Diophantus gave the method of solution of a quadratic equation which is known as "completing the square," and also solved a cubic equation equivalent to $x^3 + x = 4x^2 + 4$ in our notation.

In addition to his work on quadratic equations, Diophantus was the first to introduce algebraic symbols in the modern sense and did pioneer work on indeterminate equations, which are known after him as *Diophantine equations*.

The general solution of all cases of quadratic equations seems to have been given first about 530 A.D. by a mathematician of India named Arya-Bhatta.

The name *algebra*, as mentioned above, came from the Arabians, the word "al-gebr" being part of the title of a work, "Al-gebr we Al-mukubala," by an Arabian named Al-Khwārizmī, who lived in the first part of the eighth century A.D. The word "al-gebr" means, as stated, the "addition or subtraction of the same quantity on both sides of an equation" and "al-mukubala" means the "conbination of like terms into a single term." Al-Khwārizmī introduced the terms *root* and *power* into algebra and gave rules for solving quadratic equations. He was the first to recognize that a quadratic equation has two roots but he ignored negative and imaginary roots.

The work of Al-Khwārizmī was continued by the brilliant and accomplished Arabian scholar Tabit ibn Karra, who lived 836–901 A.D. He published a work on algebra which contained a chapter on the solution of cubic equations by geometrical methods but did not arrive at a complete general solution.

A general solution of the cubic equation was given by another

Arabian, Al-Karki, about 1000 A.D. in a book which was translated into French in 1853 and into German in 1878.

The solution of the cubic equation which is the one now generally given in books on algebra and which is the most simple and direct, was discovered by an Italian named Nicholas Tartaglia (1500–1559), but it was first made public by another Italian named Girolamo Cardan (1501–1576) and is sometimes known as *Cardan's Solution.* A pupil of Cardan named Ludovico Ferrari (1522–1562) was the first to give a general solution of the quartic equation, and it is his solution which is the one now generally given in books on algebra. The peculiar circumstances surrounding the discoverers and discoveries of these solutions have a curious and interesting history.

The name *Tartaglia* means "stammerer" and was given to Nicholas Tartaglia because of a defect in his speech which resulted from a terrible misfortune which befell him when he was twelve years old. His skull, palate and both jaws were cut open during the battle in which his native town was captured from the Italians by the French and in which his father was killed. Nicholas was left for dead but was later dragged to the cathedral of the town by his mother and finally nursed back to life and health by her. He grew up in the direst poverty but studied mathematics without a teacher and became a professor at Venice. He became famous through a contest at problem-solving which he won by solving by his newly discovered method, cubic equations proposed by one Fiori, while his adversary Fiori solved not a single problem proposed by Tartaglia.

This triumph attracted the attention of Cardan who invited Tartaglia to Milan with the false promise that he was to have an audience with a nobleman who wished to discuss mathematics with him. When Tartaglia found that the story of the nobleman was false he was greatly disappointed but, urged by Cardan, he gave Cardan his method of solution of the cubic, under vow of secrecy. In spite of his pledge, however, Cardan later published the method as his own.

This episode was typical of the character of Girolamo Cardan. He was born of doubtful parentage and became a drunkard, gambler, thief and apparently a murderer. In a fit of rage he is said to have cut off the ears of his younger son for some offense. This youth was a worthy son of his father, and an older son was hanged for poisoning his wife. In spite of his evil reputation, however, Cardan became a great mathematician and a famous teacher. Besides mathematics he

studied physics and astrology and gained a reputation as an astrologer. He was put in jail for casting the horoscope of Jesus Christ but was released without punishment through the influence of the Pope. He even predicted the day of his own death, and when he found himself alive at the end of the day he secretly committed suicide to uphold his reputation as a fortune teller.

Although Cardan obtained Tartaglia's solution of the cubic equation under pledge of secrecy he immediately began teaching it to his students, among whom was Ludovico Ferrari. In 1545 Cardan published a great and famous book on algebra with the title "Ars Magna." This book contained Tartaglia's method and when Tartaglia saw it, he was naturally hurt and aroused and he challenged Cardan to a public mathematical contest, hoping to publicly defeat and denounce him. Cardan accepted the challenge but when the day arrived he failed to appear. He sent in his place Ferrari, who was his most able student. Tartaglia easily defeated Ferrari but was set upon by Cardan's friends and barely escaped with his life. Ferrari, however, took no part in the attack.

Ferrari was originally employed by Cardan as a kitchen boy but when he showed interest and aptitude in algebra he was allowed by Cardan to attend his classes. He became Cardan's most brilliant pupil and became famous through his solution of the quartic equation and later as a teacher of mathematics. His solution is the one most used now, although a somewhat similar solution was given by a French mathematician named François Vieta (1540–1603). Ferrari was later poisoned by his sister.

The quintic equation and equations of higher degree have never been solved by algebraic methods and the algebraic solution of the quintic has been proven by the Norwegian mathematician Neils Abel (1802–1829) to be impossible, though Abel obtained a solution of the quintic by a method developed in the *theory of elliptic functions*, a branch of higher mathematics. This proof was later discussed and extended by Sir William R. Hamilton (1805–1865), a very famous Scotch mathematician and mathematical physicist.

The solutions of quadratic, cubic and quartic equations referred to above, and given in the following chapters, apply to equations with either literal or numerical coefficients and are exact. These and equations of higher degree may be solved by an approximate method applicable only to equations with numerical coefficients, which was

given in 1819 by W. G. Horner, an English mathematician. This method is usually referred to as *Horner's Method*, and is explained in some college textbooks and advanced high school texts. Horner's method in its usual form gives only real roots, however, and if an equation has roots which are complex numbers (Chap. 8) some other method must be used. A method much used by engineers and other scientists for solving equations of higher degree is known as the *Graeffe Method*. This method was first discovered by a Belgian named Germinal Dandelin in 1826 but he did not publicize it and it was later (1837) rediscovered and made public by the German mathematician whose name it bears, Carl H. Graeffe. This method is now explained in books on college algebra and in some engineering textbooks.

Chapter 11

SIMPLE EQUATIONS

59. Solution of a Simple Equation. If we are given the equation $2x = 4$, that is, x is such a number that when it is multiplied by 2, the product is 4, we know at once that x is 2. That is, $x = 2$ is the solution of the equation $2x = 4$. This is, of course, seen at a glance. Formally, however, we may say that since both sides of the equation may be divided by any number, we may divide both members of the equation, that is, divide the equation, by 2. Then

$$\frac{2x}{2} = \frac{4}{2}, \quad \text{or} \quad x = 2.$$

That is, we divide the equation *by the coefficient of the unknown* quantity.

Similarly if we have the equation $4y - y = 12$ we combine like terms in the binomial on the left, obtaining $3y = 12$, and then divide the equation by the coefficient of the unknown, in this case 3. This gives

$$\frac{3y}{3} = \frac{12}{3}, \quad \text{or} \quad y = 4$$

as the solution of the equation. This is generally written

$$3y = 12, \quad y = \tfrac{12}{3} = 4.$$

In both these cases, in the first to begin with and in the second by combining terms, each member is made to consist of only one term and the term containing the unknown is the first member of the equation while the second member does not contain the unknown.

Consider equation (1) of article 55,

$$4x + 1 = 8x - 7.$$

Here there are terms in both members which contain the unknown quantity x. We may, however, *transpose* the terms so that the equation becomes

$$4x - 8x = -7 - 1,$$

or, combining the terms algebraically,

$$-4x = -8.$$

If we now *divide by the coefficient of the unknown* we have

$$\frac{-4x}{-4} = \frac{-8}{-4},$$

and remembering the rule of signs for division we get finally

$$x = 2$$

as the solution of the equation.

As a final example let us solve the equation

$$2 - 3y + 6 = 8y - 4 - 15y.$$

Following the suggestion of the previous example, let us transpose the terms so that all terms containing the unknown y are in the first member and all other terms in the second. This gives

$$-3y - 8y + 15y = -4 - 2 - 6.$$

Combining terms,

$$+4y = -12.$$

Dividing by $+4$,

$$y = \frac{-12}{+4} = -3.$$

Therefore, $y = -3$ is the solution of the given equation.

In order to verify the solution the value -3 is substituted in the original equation in the place of y. This gives

$$2 - 3(-3) + 6 = 8(-3) - 4 - 15(-3),$$

or,

$$2 + 9 + 6 = -24 - 4 + 45$$

$$17 = 17.$$

This result is an identity and the solution is, therefore, correct.

If we had obtained some such result as $17 = 20$, or any two different numbers on opposite sides of the equality sign, we would have known that our result was not correct.

The procedure and results of the preceding illustrations can now be summarized, and we have for solving simple equations the following

RULE: (1) *Transpose the equation so that all terms which contain the unknown quantity are in the first member, and all other terms are in the second member.*

(2) *Combine all terms in each member algebraically, the left member then consisting of one term containing the unknown and its coefficient.*

(3) *Divide the equation by the coefficient of the unknown. The quotient on the right side is the desired value of the unknown.*

Let us now apply this rule to solve the simple equation

$$4x - (3x + 2) + 7 = x + (7 - 12x + 28).$$

Remove parentheses,

$$4x - 3x - 2 + 7 = x + 7 - 12x + 28.$$

Transpose,

$$4x - 3x - x + 12x = 7 + 28 - 7 + 2.$$

Combine,

$$12x = 30.$$

Divide by 12,

$$x = 2\tfrac{1}{2}.$$

When numbers are represented by letters the letters of the first part of the alphabet are generally used to represent known and constant numbers, the last to represent unknowns and variables. Thus if a, b, c and x or y appear in an equation the x or y is the unknown and a, b, c are considered as "knowns" even though their numerical values may not actually be specified. If, however, an equation contains only one letter, that one is the unknown no matter what it is.

Let us solve the simple equation

$$-7a + 5x + 3 = x + a - 4b + 3.$$

Transpose,

$$5x - x = a + 7a - 4b + 3 - 3.$$

Combine,

$$4x = 8a - 4b.$$

Factor,

$$4x = 4(2a - b).$$

Divide by 4,

$$x = 2a - b.$$

The equation is said to be solved for x *in terms of a and b*, the a and b being considered as known and the x as the unknown.

60. Fractional Equations and Clearing Fractions. In all the equations so far considered, the numbers appearing have been whole numbers and the coefficients of the unknowns were whole numbers. These may just as well be fractions. Thus $\frac{1}{2}z = 5$ is a simple equa-

tion with the fraction $\frac{1}{2}$ as the coefficient of the unknown z. The solution is of course $z = 10$. This may be found as usual by dividing the equation by the coefficient $\frac{1}{2}$, or by multiplying by 2, since division by a fraction is the same as multiplication by the inverted fraction.

Suppose we have the equation

$$5y + 5 = 20 + 2\tfrac{1}{2}y.$$

Transpose,

$$5y - 2\tfrac{1}{2}y = 20 - 5.$$

Combine,

$$2\tfrac{1}{2}y = 15.$$

Divide by $2\frac{1}{2}$,

$$y = 6.$$

After getting $2\frac{1}{2}y = 15$ we may also proceed as follows: Express both sides with common denominator. This is 2 and gives

$$\frac{5y}{2} = \frac{30}{2}.$$

Multiply by 2,

$$5y = 30.$$

Divide by 5,

$$y = 6,$$

as before.

In transforming the equation from the form $2\frac{1}{2}y = 15$ to the form $5y = 30$ we are said to *clear the equation of fractions*. When cleared of fractions, it is said to have integral coefficients, but when it contains fractions, it is a *fractional* equation.

The original equation

$$5y + 5 = 20 + 2\tfrac{1}{2}y$$

may be cleared of fractions before transposing as well as afterwards. Thus if we express both sides with the common denominator 2 it is

$$\frac{10}{2}y + \frac{10}{2} = \frac{40}{2} + \frac{5}{2}y,$$

or,

$$\frac{10y}{2} + \frac{10}{2} = \frac{40}{2} + \frac{5y}{2}.$$

Multiply the equation by 2 (every term); this gives

$$10y + 10 = 40 + 5y.$$

Transpose,

$$10y - 5y = 40 - 10.$$

Combine,

$$5y = 30.$$

Divide by 5,

$$y = 6,$$

as before.

Therefore, in order to solve a fractional equation, we may, in general, first clear the equation of fractions and then proceed according to the rule of the preceding article. In order to clear an equation of fractions, first *express all terms with a common denominator*, and then *multiply the equation by the denominator*. This causes the denominator to cancel and leaves the equation without fractions.

As an example, solve

$$4\tfrac{1}{3} + \frac{7x}{2} - 5 = 3\tfrac{1}{2} - \frac{x}{3} + 3\tfrac{1}{6} + \tfrac{1}{6}x.$$

The least common denominator is 6. Therefore,

$$\frac{26}{6} + \frac{21x}{6} - \frac{30}{6} = \frac{21}{6} - \frac{2x}{6} + \frac{19}{6} + \frac{x}{6}.$$

Multiply by 6,

$$26 + 21x - 30 = 21 - 2x + 19 + x.$$

Transpose,

$$21x + 2x - x = 21 + 19 + 30 - 26.$$

Combine,

$$22x = 44.$$

Divide by 22,

$$x = 2.$$

Suppose we have an equation like

$$2x + \frac{2x - 3}{4} - 5 = 8 - \frac{7x + 2}{12}.$$

This is also a fractional equation and the least common denominator is 12, but the fractions have binomial numerators. In the familiar manner, however, we may handle these binomials as single quantities, as indeed they are, for x has some particular value and when this value is multiplied by the coefficient of x and combined with the other term of the binomial the result is a single number. The minus sign

before each fraction refers to the value of the fraction *as a whole* and not to numerator or denominator separately. Therefore, we can write the equation in the form

$$2x + \frac{(2x - 3)}{4} - 5 = 8 - \frac{(7x + 2)}{12}.$$

Writing this with L.C.D. 12,

$$\frac{24x}{12} + \frac{3(2x - 3)}{12} - \frac{60}{12} = \frac{96}{12} - \frac{(7x + 2)}{12}.$$

Multiply by 12,

$$24x + 3(2x - 3) - 60 = 96 - (7x + 2).$$

Remove parentheses,

$$24x + 6x - 9 - 60 = 96 - 7x - 2.$$

Transpose,

$$24x + 6x + 7x = 96 - 2 + 60 + 9.$$

Combine,

$$37x = 163.$$

Divide by 37,

$$x = \tfrac{163}{37} = 4\tfrac{15}{37}.$$

It is to be noted that in clearing this equation of fractions the signs of the terms in the numerator of the fraction on the right member are changed, but those in the numerator of the fraction on the left remain unchanged, and that the fraction on the right is preceded by a minus sign while that on the left is preceded by plus. Thus, in clearing fractions *the sign of every term in the numerator of a fraction preceded by a minus sign is changed.* The fraction line thus has, in this respect, the same effect as parentheses, grouping the numerator terms.

61. Exercises.

Solve the following equations and verify the results:

1. $3x + 15 = x + 25.$
2. $2x - 3 = 3x - 7.$
3. $3x + 4 = 5(x - 2).$
4. $8(y - 1) + 4(y - 3) = 4(4y - 9) + 4.$
5. $5x - 6(x - 5) = 2(x + 5) + 5(x - 4).$
6. $y - [3 + \{y - (3 + y)\}] = 5.$
7. $(z + 1)(2z + 1) = (z + 3)(2z + 3) - 14.$
8. $2u - 5[3u - 7(4u - 9)] = 66.$

9. $25T - 19 - [3 - (4T - 15)] = 3T - (6T - 5).$

10. $2x + 3 = 16 - (2x - 3).$

11. $\dfrac{4(x - 2)}{5} = 7 + \dfrac{5x}{11}.$

12. $\dfrac{x + 4}{14} + \dfrac{x - 4}{6} = 2.$

13. $\dfrac{y + 20}{9} = 6 - \dfrac{3y}{7}.$

14. $\dfrac{z - 8}{7} + \dfrac{z - 3}{5} + \dfrac{5}{2} = 0.$

15. $\dfrac{5(x - 5)}{8} - \dfrac{2(x - 3)}{7} = 5\tfrac{19}{28}.$

Chapter 12

SIMULTANEOUS SIMPLE EQUATIONS

62. Equations with Two Unknowns. To beginners it often seems that a simple equation containing two unknowns should be solved for either one of them by proceeding in the same way as in the case of a single unknown. Thus, if the equation is

$$3x - 2y = 5$$

we would according to that method write

$$3x = 5 + 2y,$$

and hence

$$x = \frac{5 + 2y}{3}.$$

There is no error in this algebraic transformation, and this equation for x is correct in form, but it does not give the value of x because y *is not known* and, therefore, the values of $5 - 2y$ and $(5 - 2y)/3$ are not known. In order to find x, therefore, we must in some way eliminate y from the equation so as to obtain an equation *containing x alone*, or eliminate x to obtain an equation containing y alone. When this is done, either x or y may be found in the usual way. There are several ways of doing this.

We have seen that, in order to solve equations for *two unknowns*, we must have *two equations*. Suppose the unknowns are x and y and the equations are the one given above,

together with

$$\left.\begin{aligned} 3x - 2y &= 5, \\ 4x + 2y &= 9. \end{aligned}\right\} \tag{2}$$

From the first, as above, we have

$$x = \frac{5 + 2y}{3}, \tag{3}$$

and from the second,

$$x = \frac{9 - 2y}{4}.$$

Now the value of x is the same in both the original equations. Therefore, the last two expressions for x must be equivalent. That is, it must be that

$$\frac{5 + 2y}{3} = \frac{9 - 2y}{4}. \tag{4}$$

This is a fractional simple equation *containing y alone*, the x having been eliminated. The x has not been discarded, however, the two original equations have only been combined into one which does not contain x. They might also have been combined so as to eliminate y.

The fractional simple equation (4) is solved in the usual way. The least common denominator is 12. Therefore, we have

$$\frac{4(5 + 2y)}{12} = \frac{3(9 - 2y)}{12}$$

Multiply by 12,

$$4(5 + 2y) = 3(9 - 2y),$$

or,

$$20 + 8y = 27 - 6y.$$

Transpose,

$$8y + 6y = 27 - 20.$$

Combine,

$$14y = 7.$$

Divide by 14,

$$y = \tfrac{1}{2}.$$

With the value of y known, the expression (3) will *now* give the value of x. For, if we put $\frac{1}{2}$ for y in (3), it gives,

$$x = \frac{5 + 1}{3} = \frac{6}{3} = 2.$$

The solutions of the two original simultaneous equations (2) are, therefore,

$$x = 2, \quad y = \tfrac{1}{2}.$$

As stated above, we might just as well have first eliminated y and solved for x. We proceed to do this. First transposing the x terms in both the equations (2), we have

$$-2y = 5 - 3x,$$

$$2y = 9 - 4x.$$

Dividing by the coefficients of y,

$$y = \frac{5 - 3x}{-2} = \frac{3x - 5}{2}$$

$$y = \frac{9 - 4x}{2}. \tag{5}$$

Placing these two expressions for y equal to one another,

$$\frac{3x - 5}{2} = \frac{9 - 4x}{2}.$$

Multiply by 2,

$$3x - 5 = 9 - 4x.$$

Transpose,

$$3x + 4x = 9 + 5.$$

Combine,

$$7x = 14.$$

Divide by 7,

$$x = 2.$$

Substitute this value of x in (5). Then,

$$y = \frac{9 - (4 \times 2)}{2} = \frac{9 - 8}{2} = \frac{1}{2}.$$

The results are the same as before, namely,

$$x = 2, \quad y = \tfrac{1}{2}.$$

From these discussions and results it is obvious that in order to solve two simultaneous equations for two unknowns, it is necessary temporarily to *eliminate one of the unknowns by combining the two equations into one equation*, which only contains one of the unknowns. This simple equation is then readily solved for that unknown in the usual way. With one of the original unknowns now known, its value can be substituted for the symbol in one of the previous equations, and from the resulting simple equation, the other unknown can be found. As already stated there are several methods of eliminating one of the unknowns and combining the two original equations into one. The method we have just used is called the method of *elimination by comparison*, from the fact that each of the equations is solved for the same

unknown in terms of the other and the two resulting expressions are compared or equated, that is, placed equal.

The entire procedure in solving two simultaneous equations by the method of *elimination by comparison* is now plain. We have, namely, the following

RULE: (1) *Solve each equation for the same unknown in terms of the other and place the two resulting expressions equal.*

(2) *Solve the resulting simple equation for the single unknown which it contains.*

(3) *Substitute the resulting value for this unknown in one of the expressions found in step* (1) *and reduce and simplify. The result is the value of the other unknown.*

(4) *Verify both results by substituting both in both the original equations.*

Another simple method of elimination of one of the unknowns is the method of *addition and subtraction.* It depends on the property of equations which says that if equals are added or subtracted the results are equal (article 56). The following solution will illustrate and explain the method.

Take the same equations already solved by another method,

$$3x - 2y = 5,$$

$$4x + 2y = 9.$$

Since each of the left members is equal to its right member, the algebraic sum of the left members is equal to the algebraic sum of the right members, according to the property of equations just stated. Therefore if we add the members separately, remembering the rules of signs, we have

$$\begin{array}{rcr} 3x - 2y & = & 5 \\ 4x + 2y & = & 9 \\ \hline 7x \qquad & = & 14 \end{array}$$

$$\therefore \quad x = 2.$$

Substituting this value of x in one of the original equations, say the second, we have

$$(4 \times 2) + 2y = 9.$$

Transpose,

$$2y = 9 - 8 = 1$$

$$\therefore \quad y = \tfrac{1}{2}.$$

These results are the same as those obtained by the other method.

Let us now apply this method to the following equations:

$$4x + 3y = 8$$

$$x - 2y = 13.$$

Neither x nor y is made to disappear by adding these equations as they stand. If, however, the first equation is multiplied by 2 and the second by 3, they become

$$\begin{array}{r} 8x + 6y = 16 \\ 3x - 6y = 39 \\ \hline 11x \qquad = 55 \end{array}$$

and on adding,

$$\therefore \quad x = 5.$$

Substituting this value of x in the first of the original equations,

$$20 + 3y = 8.$$

Transpose,

$$3y = 8 - 20 = -12.$$

Divide by 3,

$$y = -4.$$

The solutions are, therefore, $x = 5$ and $y = -4$.

Consider next the equations

$$4x + 10y = 12$$

$$2x + 3y = 2.$$

Multiply the second equation by 2. They are, then,

$$4x + 10y = 12,$$

$$4x + 6y = 4.$$

Neither x nor y is eliminated by adding the equations. If, however, we *subtract* the second from the first, the x terms cancel. Carrying out the subtraction, we have

$$\begin{array}{r} 4x + 10y = 12 \\ 4x + 6y = 4 \\ \hline 4y = 8 \end{array}$$

$$\therefore \quad y = 2.$$

This value of y in one of the original equations, say the second, gives,

$$2x + (3 \times 2) = 2.$$

Transpose, $$2x = 2 - 6 = -4.$$

$$\therefore \quad x = -2.$$

The solutions are, therefore, $x = -2$ and $y = 2$.

These examples are sufficient to show that two simultaneous simple equations with two unknowns are solved by the *method of addition and subtraction* according to the following

RULE: (1) *Clear of fractions and transpose both equations so that all terms containing unknowns are in the left members and all other terms are on the right.*

(2) *If neither unknown has the same coefficient in both equations, multiply one or both equations so that the coefficient of one unknown is the same in both equations.*

(3) *Add or subtract the equations member by member so that one of the unknowns is eliminated, and place the results equal.*

(4) *Solve the resulting equation for the unknown which it contains.*

(5) *Place the resulting value in one of the original equations and solve this equation for the other unknown.*

There is a third method of elimination which is sometimes used, but it is usually explained whenever it is used and will not be given here.

63. Equations with Three Unknowns. In solving two equations for two unknowns, the procedure was to combine the two equations into one which contains one unknown. This equation we already knew how to solve.

Similarly in the case of three equations with three unknowns, such as

$$3x - 2y + 4z = 13 \tag{6}$$

$$2x + 5y - 3z = -9 \tag{7}$$

$$6x + 3y + 2z = 7 \tag{8}$$

the following procedure suggests itself: Combine two of the equations into one, eliminating one unknown; combine another pair of the equations, eliminating the same unknown. There are then two equations containing two unknowns. These two may then be solved by either of the methods of the preceding article. With these two of the original unknowns now known, their values can be substituted in one of the original equations and an equation containing the third unknown obtained. Its solution for this unknown completes the solution of the three original equations. Let us try this procedure with the three equations given. Any one of the unknowns may be eliminated first.

Suppose we eliminate z from the pair of equations (6) and (8). Multiply equation (8) by 2. This gives,

$$12x + 6y + 4z = 14$$

Rewrite (6),

$$\begin{array}{rl} 3x - 2y + 4z &= 13 \\ \hline 9x + 8y \qquad &= 1. \end{array}$$

Subtract, $\qquad 9x + 8y = 1. \qquad (9)$

Next eliminate z from the pair (6) and (7), for example. Multiply (6) by 3 and (7) by 4. This gives

$$\begin{array}{rl} 9x - 6y + 12z &= 39 \\ 8x + 20y - 12z &= -36 \\ \hline 17x + 14y \qquad\quad &= 3. \end{array}$$

Add, $\qquad 17x + 14y = 3. \qquad (10)$

The two equations (9) and (10) contain only two unknowns, x and y. They are solved as in the preceding article by either of the methods of elimination. We will use the method of subtraction and eliminate y. To do this, multiply equation (10) by 4 and (9) by 7. They then become

$$\begin{array}{rl} 68x + 56y &= 12 \\ 63x + 56y &= 7 \\ \hline 5x \qquad\quad &= 5. \end{array}$$

subtract, $\qquad 5x = 5.$

$$\therefore \quad x = 1.$$

Substitute this value of x in one of the originals of these two equations, say (9). This gives

$$9 + 8y = 1.$$

Transpose,

$$8y = 1 - 9 = -8$$

$$\therefore \quad y = -1.$$

Substitute these values of x and y in one of the original three equations, say (8). This gives

$$6 - 3 + 2z = 7.$$

Transpose,

$$2z = 7 + 3 - 6 = 4.$$

$$\therefore \quad z = 2.$$

The solutions of the three simultaneous equations (6), (7) and (8) are therefore $x = 1$, $y = -1$, $z = 2$.

In eliminating the first unknown and combining the three equations into two with two unknowns, the method of addition and subtraction will usually be found more convenient. In solving the two equations resulting from the first elimination, either of the methods of article 62 may be used.

The example just worked out is sufficient illustration of the method of solution of *three simultaneous equations with three unknowns.* This method is summarized in the following

RULE: (1) *Clear the equations of fractions and transpose so that all terms containing the unknowns are in the first members and all other terms on the right.*

(2) *Eliminate one of the unknowns from two of the equations by addition or subtraction, obtaining one equation containing two unknowns.*

(3) *Eliminate the same unknown from a different pair of the original equations, obtaining a second equation containing two unknowns.*

(4) *Solve the two simultaneous equations obtained in steps* (2) *and* (3) *for the two unknowns they contain.*

(5) *Substitute the two values obtained in step* (4) *in one of the original equations and solve for the third unknown.*

64. Equations with More than Three Unknowns. It is now apparent that in the case of four simultaneous simple equations with four unknowns, the same unknown is to be eliminated from three different pairs of the equations, three new equations with three unknowns being obtained. These three are then to be solved as in the preceding article.

Similarly, in the case of five equations, and so on, the number of equations is reduced one at a time by eliminating the unknowns one at a time in succession. With as many as four or more unknowns, however, this method becomes very laborious, and in more advanced books on algebra, special condensed methods are developed which depend on a special principle, the theory of *determinants.* When the reader reaches the point where such *systems* of equations have to be solved (as in electric circuit theory, advanced mechanics, etc.) he will usually have studied determinants.

For these reasons, and also because no new principle is learned in extending to such systems of equations the elimination methods developed above, the subject will not be pursued farther here.

65. Exercises.

Solve the following equations by any method and verify the solutions:

1. $3x + 4y = 10.$
   $4x + y = 9.$
2. $x + 2y = 13.$
   $3x + y = 14.$
3. $5x = 7y - 21.$
   $21x - 75 = 9y.$
4. $2u - v = 9.$
   $3u - 7v = 19.$
5. $\dfrac{2x}{3} + y = 16.$
   $x + \dfrac{y}{4} = 14.$
6. $\dfrac{x}{5} + \dfrac{y}{2} = 5.$
   $x - y = 4.$
7. $\dfrac{z}{9} + \dfrac{y}{7} = 10.$
   $\dfrac{z}{3} + y = 50.$
8. $x + 2y + 2z = 11.$
   $2x + y + z = 7.$
   $3x + 4y + z = 14.$
9. $x + 3y + 4z = 14.$
   $x + 2y + z = 7.$
   $2x + y + 2z = 2.$
10. $3x - 4y = 6z - 16.$
    $4x - y = z + 5.$
    $3y + 2(z - 1) = x.$
11. $6x + 2y - 5z = 13.$
    $3x + 3y - 2z = 13.$
    $7x + 5y - 3z = 26.$
12. $5x + 2y = 14.$
    $y - 6z = -15.$
    $x + 2y = -z.$
13. $2x + 3y + 4z = 20.$
    $3x + 4y + 5z = 26.$
    $3x + 5y + 6z = 31.$
14. $2x - y + 3z - 2u = 14.$
    $x + 7y + z - u = 13.$
    $3x + 5y - 5z + 3u = 11.$
    $4x - 3y + 2z - u = 21.$
15. $u + v + w = 7.$
    $v + w + x = -8.$
    $w + x + u = 5.$
    $x + u + v = -10.$

Chapter 13

QUADRATIC EQUATIONS

66. Forms of Quadratic Equations. Both the equations

$$4x^2 - 75 = x^2 \tag{11}$$

and

$$3x^2 - 7x = 2x^2 - 12 \tag{12}$$

are quadratic equations, but they differ in that the first contains only the second power of the unknown, x, while the second contains both the first and second powers of the unknown.

An equation which contains the second power alone of the unknown is called a *pure quadratic* equation and one which contains both the first and second powers is called a *complete quadratic.* Either pure or complete quadratics may be integral or fractional.

A pure quadratic equation is very easily solved. Thus when the terms in equation (11) are transposed it becomes

$$4x^2 - x^2 = 75.$$

Combining terms,

$$3x^2 = 75.$$

Divide by 3,

$$x^2 = 25.$$

Now since the square of x is 25 we know at once that x is either $+5$ or -5, or, as it is usually written, ± 5. Otherwise expressed we may take the square root of both sides of the last equation (see article 56) and write

$$\sqrt{x^2} = \sqrt{25}$$

$$\therefore \quad x = \pm 5.$$

The above procedure makes it plain that in order to solve any pure quadratic equation: Transpose all terms containing the unknown to the first member and all others to the second; combine terms; divide the equation by the coefficient of the square of the unknown; take the square root of both sides of the resulting equation, placing the double

sign before the result on the right. These are the required values of the unknown quantity.

Since it is only necessary to deal with squares in solving a pure quadratic equation the solution is obtained by taking square roots. In the complete quadratic equation, however, there is a first power as well as a square and, therefore, the square root is not to be taken immediately. We shall see later that by a process of completing the square, so as to obtain the exact square of a binomial, square roots may be taken. Before taking up the discussion of this method, however, we shall first discuss a simpler method which applies to certain complete quadratic equations.

67. Solution of a Complete Quadratic by Factoring. Consider the second of the two equations given above, equation (12), and transpose *all* terms to the left side. When all terms are transposed (subtracted) from the right side that member is zero and the equation becomes

$$3x^2 - 7x - 2x^2 + 12 = 0.$$

Combining like terms,

$$x^2 - 7x + 12 = 0. \tag{13}$$

The first member is now a trinomial which can be factored.

$$(x - 4)(x - 3) = 0.$$

This equation now states that the product of the binomials $x - 3$ and $x - 4$ is zero.

Now, when the product of any two numbers is zero, either one or the other is zero, or both are zero. (For example $2 \times 0 = 0$, $0 \times a = 0$, $0 \times 0 = 0$, etc.) Therefore, in the last equation above, either the factor $(x - 3)$ or the factor $(x - 4)$ or both, must be zero. The equation is satisfied, that is, the product is zero, if either factor is zero. Therefore, we can write

$$x - 3 = 0, \quad \text{and} \quad x - 4 = 0.$$

These are two simple equations and both are solved at once by transposing. This gives

$$x = 3, \quad x = 4. \tag{14}$$

The condition expressed by the quadratic equation (12), or (13), is therefore satisfied by either of the values 3 or 4 for the unknown x, and the two equations (14) represent *solutions* of the quadratic equation (12) or (13).

The solutions of a quadratic equation are verified by substituting each separately in turn in the original equation. Thus if the value 3 be substituted for x in (13) we have

$$(3)^2 - 7(3) + 12$$
$$= 9 - 21 + 12 = 0,$$

which is an identity and therefore verifies the solution $x = 3$. Similarly, using the value 4 for x we have

$$(4)^2 - 7(4) + 12$$
$$= 16 - 28 + 12 = 0,$$

which is also an identity and verifies the solution $x = 4$.

The method of solution of a complete quadratic equation which we have used here is called *solution by factoring.* Since any complete quadratic may be reduced to the form corresponding to (13) by transposing and combining like terms, any quadratic equation may be solved if its transposed form, with right member zero, is a *factorable trinomial.*

Let us try the equation

$$2x^2 - 10x + 27 = 8x - x^2. \tag{15}$$

Transpose, $2x^2 + x^2 - 10x - 8x + 27 = 0.$

Combine, $3x^2 - 18x + 27 = 0.$

Divide by 3, $x^2 - 6x + 9 = 0.$ (16)

Factor, $(x - 3)(x - 3) = 0.$

Therefore, $x - 3 = 0, \quad x - 3 = 0.$

$$\therefore \quad x = 3, \qquad x = 3,$$

are the solutions of equation (15) or (16).

As a final example, let us solve the equation

$$6x^2 - 3x - 30 = 0. \tag{17}$$

Factor, $(3x + 6)(2x - 5) = 0.$

Therefore, $3x + 6 = 0, \quad 2x - 5 = 0.$

Transpose, $3x = -6, \qquad 2x = 5.$

$$\therefore \quad x = -2, \qquad x = 2\tfrac{1}{2},$$

are the solutions of (17).

If the trinomial which results from transposing all terms to the left member and combining like terms is not factorable, the quadratic cannot be solved by this method.

The preceding examples are sufficient to show that a quadratic is *solved by factoring* according to the following

RULE: (1) *Transpose all terms to the left member, leaving the right member zero, and combine like terms on the left.*

(2) *If the terms have a common monomial factor divide the equation by this factor.*

(3) IF THE RESULTING TRINOMIAL IS FACTORABLE *separate it into two binomial factors and place each equal to zero, giving two simple equations.*

(4) *Solve each of these simple equations for the unknown; the results are the solutions of the original quadratic equation.*

68. The Roots of a Quadratic Equation. It is probably already noted that each of the quadratic equations solved in the preceding two articles has two solutions. Thus equation (11) is satisfied by the values $+5$ and -5, equation (12) by 3 and 4, equation (15) by 3 and 3, and (17) by -2 and $+2\frac{1}{2}$. In the first case the double result is obviously due to the fact that the unknown x is found to be equal to $\sqrt{25}$, and as we have already seen, the square root of a positive number may be either positive or negative. In the other cases, this is not so obvious as no square roots are extracted, but it is still true in each case that in the beginning we had the square of the unknown and in the end the first power, so that some process equivalent to taking the square root had been carried out. Thus it appears that a quadratic equation has *two* solutions.

In the first case considered above, the two roots are numerically equal but have opposite signs; in the second case both have the same sign but are not the same number; in the last case they are unequal and of opposite sign. In the third case, however, they are *equal and of the same sign.* It might seem that in this case there is only one solution, namely, $x = 3$. In all other cases, however, there are two different roots, and in this case they came out separately in the same manner as in the other cases, so that even in this case the equation is said to have two roots or solutions. Therefore, a quadratic equation *always has two roots.*

When the two roots of a quadratic equation are equal and of the same sign, they are called *duplicate roots.* Otherwise they are said to be *distinct roots.*

We have seen that a simple equation only has one root, and we shall see later that a cubic equation has three roots. Similarly, a quartic equation has four roots, and, in general, the number of roots of an equation is the same as the degree of the equation.

69. Solution by Completing the Square. The following quadratic equation, in the transposed form, cannot be factored:

$$x^2 + 4x - 7 = 0.$$

In this case the rule given in article 67 does not apply since the left side of the equation cannot be factored, and a different procedure is necessary. Let us transpose to the right member the term which does not contain the unknown x. This gives

$$x^2 + 4x = 7. \tag{18}$$

Now, from the results of article 17, it is known that $x^2 + 4x + 4$ is the square of the binomial $x + 2$. If, therefore, 4 be added to the left member of (18), that member will be the exact square of $(x + 2)$. If 4 is added to the left member, however, 4 must be added to the right member in order to preserve the equality, according to article 56. Adding 4 to both sides of equation (18) it becomes

$$x^2 + 4x + 4 = 11,$$

or

$$(x + 2)^2 = 11.$$

Taking the square root of both sides there results

$$x + 2 = \pm\sqrt{11}.$$

This is equivalent to two equations, namely,

$$x + 2 = +\sqrt{11},$$
$$x + 2 = -\sqrt{11}.$$

Transposing the $+2$ in both of these simple equations,

$$x = -2 + \sqrt{11},$$
$$x = -2 - \sqrt{11}.$$

Since $\sqrt{11} = 3.317$ approximately, the roots may be written as $x = 1.317 \ldots$ and $x = -5.317 \ldots$ and are the roots of the quadratic (18).

It is to be noted here that the original quadratic equation was transposed so that only those terms containing the unknown are on

the left and the term which does not contain it is on the right; the coefficient of the square term (x^2) is 1; and to *both sides* of the resulting equation is added the number which makes the left member the *exact square of a binomial.* The equation then states that the *square* of a binomial is equal to a certain number, and the rest is easy. The square root of both sides is taken as in solving a pure quadratic. This method of solving a quadratic equation is called *completing the square.* It applies to any complete quadratic equation, whether it is factorable or not.

As another example let us apply this method to the equation (13) above which was solved by factoring. The equation is

$$x^2 - 7x + 12 = 0.$$

Transposing,

$$x^2 - 7x = -12. \tag{19}$$

Now in order for a trinomial to be the exact square of a binomial of the form $(x - a)^2 = x^2 - 2ax + a^2$, the coefficient of the first power term (x) in the square is twice the second (constant) term in the binomial, and the last term in the square is the square of this number, that is, it is the square of *half the coefficient of x.* Therefore, in order to have an exact square on the left of (19), there must be added to that member the square of half the coefficient 7, that is, $(\frac{7}{2})^2$ or $\frac{49}{4}$, and this must also be added to the right member to preserve the equality.

Adding this number to both sides of (19), we have

$$x^2 - 7x + \tfrac{49}{4} = -12 + \tfrac{49}{4} = \tfrac{1}{4}.$$

The left member is now the square of $x - \frac{7}{2}$. The equation is, therefore,

$$(x - \tfrac{7}{2})^2 = \tfrac{1}{4}.$$

$$\therefore \quad x - \tfrac{7}{2} = \sqrt{\tfrac{1}{4}} = \pm\tfrac{1}{2},$$

or,

$$x - \tfrac{7}{2} = +\tfrac{1}{2},$$

and

$$x - \tfrac{7}{2} = -\tfrac{1}{2}.$$

Transposing these two simple equations,

$$x = \tfrac{7}{2} + \tfrac{1}{2}, \quad x = \tfrac{7}{2} - \tfrac{1}{2}.$$

$$\therefore \quad x = \tfrac{8}{2} = 4,$$

$$x = \tfrac{6}{2} = 3,$$

and these solutions are the same as those obtained before by factoring, see equations (14).

In this example, as in the preceding one, the coefficient of the square term is 1. If it is not equal to 1, the coefficient of x is not twice the second term of the binomial. In such a case, however, the coefficient of the square term can be made equal to 1 by dividing the equation by the actual coefficient.

Thus, if the equation

$$2x^2 - 3x = 2 \tag{20}$$

is divided by 2, it becomes

$$x^2 - \tfrac{3}{2}x = 1.$$

The coefficient of x^2 is now 1, and that of x is $\frac{3}{2}$. Half this coefficient is $\frac{1}{2}(\frac{3}{2}) = \frac{3}{4}$ and the square of $\frac{3}{4}$ is $(\frac{3}{4})^2 = \frac{9}{16}$. Adding this *square of half the coefficient of* x to both sides of the equation, it becomes,

$$x^2 - \tfrac{3}{2}x + (\tfrac{3}{4})^2 = 1 + \tfrac{9}{16},$$

or,

$$x^2 - \tfrac{3}{2}x + (\tfrac{3}{4})^2 = \tfrac{25}{16}.$$

The left member is now the square of the binomial $(x - \frac{3}{4})$, so the equation is,

$$(x - \tfrac{3}{4})^2 = \tfrac{25}{16}$$

$$\therefore \quad x - \tfrac{3}{4} = \sqrt{\tfrac{25}{16}} = \pm\tfrac{5}{4}.$$

Hence,

$$x - \tfrac{3}{4} = +\tfrac{5}{4}, \quad \text{or} \quad x - \tfrac{3}{4} = -\tfrac{5}{4},$$

and

$$x = \tfrac{5}{4} + \tfrac{3}{4}, \quad x = -\tfrac{5}{4} + \tfrac{3}{4}.$$

$$\therefore \quad x = 2, \qquad x = -\tfrac{1}{2}$$

are the roots of the quadratic equation (20).

From these examples and results, we can now state the following as the method of solving quadratic equations by *completing the square:*

RULE: (1) *Clear the equation of fractions, transpose all terms containing the unknown to the left member of the equation and all others to the right, and combine like terms.*

(2) *If the coefficient of the square term is not equal to* 1 *divide the equation by that coefficient and reduce all resulting fractions to lowest terms.*

(3) *Add to both sides of the equation the square of half the coefficient of the*

first power of the unknown, indicating the addition on the left and performing it on the right.

(4) *The left member is now the square of a binomial. Express it in this form, the first term of the binomial being the unknown, the second term the number whose square was added in step* (3), *and its sign that of the first power of the unknown in the equation.*

(5) *Write the binomial of step* (4) *equal to the square root of the right member of the equation, with double sign, and solve the two resulting simple equations. The solutions are the roots of the quadratic.*

In order to show the workings of this rule in a complete solution, we shall apply it to solve the following equation:

$$\frac{2x}{3} + \frac{3x^2 - 4}{2} = \frac{1}{2} + \frac{5}{3}x.$$

The least common denominator is 6. Therefore

$$\frac{4x}{6} + \frac{3(3x^2 - 4)}{6} = \frac{3}{6} + \frac{10x}{6}.$$

Multiply the equation by 6; this gives,

$$4x + 3(3x^2 - 4) = 3 + 10x,$$

or,

$$4x + 9x^2 - 12 = 3 + 10x.$$

Transpose, $\quad 4x + 9x^2 - 10x = 3 + 12.$

Combine, $\quad 9x^2 - 6x = 15.$

Divide by 9, $\quad x^2 - \frac{2}{3}x = \frac{5}{3}.$

The square of half the coefficient of x is $(\frac{1}{3})^2 = \frac{1}{9}$.

Add $(\frac{1}{3})^2$ to both sides,

$$x^2 - \frac{2}{3}x + (\tfrac{1}{3})^2 = \frac{5}{3} + \frac{1}{9} = \frac{16}{9}.$$

That is, $\quad (x - \frac{1}{3})^2 = \frac{16}{9}.$

$$\therefore \quad x - \frac{1}{3} = \sqrt{\tfrac{16}{9}} = \frac{\sqrt{16}}{\sqrt{9}} = \pm\frac{4}{3},$$

or,

$$x - \frac{1}{3} = +\frac{4}{3}, \quad x - \frac{1}{3} = -\frac{4}{3}.$$

$$\therefore \quad x = \frac{4}{3} + \frac{1}{3}, \quad x = -\frac{4}{3} + \frac{1}{3},$$

or,

$$x = \frac{5}{3}, \quad x = -1.$$

70. The Quadratic Formula. When like terms are collected and combined in a complete quadratic equation, there will, in general, be three terms: one containing the square and one the first power of the unknown, and one which does not contain the unknown. When these terms are all transposed to the left member of the equation, this member will be a trinomial and the right member will be zero. The term of the trinomial which does not contain the unknown, and also the coefficients of the other two terms, may be numerical or literal numbers, positive or negative, and they may be monomials or polynomials. The two coefficients and the third term are generally referred to collectively as the *coefficients* of the equation. The equations (13), (16), (17) of this chapter are examples of such transposed equations with numerical monomial coefficients.

If, therefore, we let A represent the coefficient of the square term, B that of the first power term, and C the third term, any complete quadratic equation can be written in the form

$$Ax^2 + Bx + C = 0, \tag{21}$$

where x represents any unknown and the coefficients A, B, C may be positive or negative, numerical or literal, monomial or polynomial. The form (21) is, therefore, called the *general form* of the complete quadratic equation. If, therefore, we can express concisely and in symbols the rule for solving equation (21), it will apply to all quadratic equations.

We now solve the equation (21) by completing the square.

Transpose,

$$Ax^2 + Bx = -C.$$

Divide by A,

$$x^2 + \frac{B}{A}x = -\frac{C}{A}.$$

Half the coefficient of x is $B/2A$ and the square of this is

$$\left(\frac{B}{2A}\right)^2 = \frac{B^2}{4A^2}.$$

Adding this to both sides of the equation,

$$x^2 + \frac{B}{A}x + \left(\frac{B}{2A}\right)^2 = \frac{B^2}{4A^2} - \frac{C}{A}.$$

Combining the two fractions in the second member with the common denominator $4A^2$, the equation becomes,

$$x^2 + \frac{B}{A}x + \left(\frac{B}{2A}\right)^2 = \frac{B^2 - 4AC}{4A^2}.$$

The left member is now the square of the binomial $x + \frac{B}{2A}$. Therefore

$$\left(x + \frac{B}{2A}\right)^2 = \frac{B^2 - 4AC}{4A^2}.$$

Taking the square root of both members,

$$x + \frac{B}{2A} = \pm\sqrt{\frac{B^2 - 4AC}{4A^2}}$$

or, taking the root of numerator and denominator separately on the right,

$$x + \frac{B}{2A} = \pm\frac{\sqrt{B^2 - 4AC}}{2A}.$$

Transposing the term $B/2A$ this gives for x,

$$x = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

and since both terms on the right have the common denominator $2A$, we have finally,

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \qquad (22)$$

In any given equation in the form (21) the signs and values of A, B and C will be known. When these are substituted in the right side of (22) and the indicated calculations are carried out, the results will be the solutions of the given equation.

An equation like (22), in which the left member is a single symbol representing a single quantity, and the right member is a combination of other symbols whose values may be known, is called a *formula*. When the values of these other symbols on the right are substituted in place of the symbols themselves and the calculations performed, the value of the symbol on the left is said to be *calculated from the formula*.

The formula (22) is called the *quadratic formula*, and when the coefficients of a quadratic equation are substituted in this formula the value of the unknown can be calculated at once. This method of calculating the roots of a quadratic equation is called *solution by formula*. It is to be noted that in the quadratic formula there are two signs

before the radical and, therefore, two values will be obtained in the calculation. These are the regular double roots of the quadratic.

71. Solution of Quadratics by Formula. In order to illustrate the use of the quadratic formula we will first solve by means of the formula one of the equations already solved by other methods, showing every step in the calculation of the roots.

Take the equation (13), article 67,

$$x^2 - 7x + 12 = 0.$$

Comparing this with

$$Ax^2 + Bx + C = 0,$$

we have,

$$A = 1, \quad B = -7, \quad C = 12.$$

Substituting these values in the formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

and reducing, we have,

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - (4 \times 1 \times 12)}}{2 \times 1} = \frac{7 \pm \sqrt{49 - 48}}{2}$$

$$= \frac{7 \pm \sqrt{1}}{2} = \frac{7 \pm 1}{2}.$$

Using the plus sign,

$$x = \tfrac{8}{2} = 4.$$

Using the minus sign,

$$x = \tfrac{6}{2} = 3.$$

These are the same as the values already found (see equations (14)).

Take next the equation (17),

$$6x^2 - 3x - 30 = 0.$$

The equation can be simplified by dividing by 3,

$$2x^2 - x - 10 = 0.$$

Here

$$A = 2, \quad B = -1, \quad C = -10,$$

and hence,

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 2 \times (-10)}}{2 \times 2}$$

$$= \frac{1 \pm \sqrt{1 + 80}}{4} = \frac{1 \pm \sqrt{81}}{4}$$

$$= \frac{1 \pm 9}{4}.$$

Using the plus sign,

$$x = \frac{10}{4} = 2\tfrac{1}{2}.$$

Using the minus sign,

$$x = \frac{-8}{4} = -2,$$

as before.

In using the quadratic formula it is to be remembered that the equation should be cleared of fractions, all terms must be transposed to the left member, and like terms must be combined. If the coefficients contain a common factor the equation should be divided by this factor. These preliminary transformations are the same in any method of solution; in using the formula, therefore, it is only necessary to take as examples equations which are already in the form (21) above.

Thus let us take the equation

$$6y^2 + 17ay + 12a^2 = 0.$$

Here y is considered as the unknown quantity and a is considered as known. We have therefore

$$A = 6, \quad B = 17a, \quad C = 12a^2.$$

$$\therefore \quad y = \frac{-17a \pm \sqrt{(17a^2) - 4 \times 6 \times 12a^2}}{2 \times 6}$$

$$= \frac{-17a \pm \sqrt{289a^2 - 288a^2}}{12}$$

$$= \frac{-17a \pm \sqrt{a^2}}{12} = -\frac{17a \pm a}{12}.$$

Using the plus sign,

$$y = \frac{-17a + a}{12} = \frac{-16a}{12} = -\frac{4a}{3}.$$

Using the minus sign,

$$y = \frac{-17a - a}{12} = \frac{-18a}{12} = -\frac{3a}{2}.$$

72. Quadratics with Complex Roots. As a final example let us solve the equation

$$x^2 - 6x + 10 = 0.$$

Here

$$A = 1, \quad B = -6, \quad C = 10.$$

and hence

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 1 \times 10}}{2 \times 1}$$

$$= \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm \sqrt{-4}}{2}.$$

Now, according to the methods of articles 41 and 45, $\sqrt{-4} = \sqrt{4 \times (-1)} = \sqrt{4} \times \sqrt{-1} = 2\sqrt{-1} = 2i$. Therefore

$$x = \frac{6 \pm 2i}{2}.$$

Using the plus sign,

$$x = 3 + i.$$

Using the minus sign,

$$x = 3 - i.$$

The roots of the given equation are, therefore, complex numbers, and since their respective real and imaginary parts are the same with opposite connecting signs, they are conjugate complex numbers.

Quadratic equations will frequently be found which have imaginary or complex roots. As the roots are both formed from the same radical the respective real and imaginary parts of the two roots, whenever complex roots occur, will always be the same, that is, complex roots are always conjugate.

The radical in the formula will give rise to an imaginary number whenever the number under the radical is negative. As this number is the binomial $B^2 - 4AC$, formed from the coefficients of the equation, it can be determined in advance whether or not the roots are complex,

without solving the equation. This is done by substituting the values of the coefficients A, B, C in the expression

$$B^2 - 4AC.$$

If this expression is then negative, the roots are conjugate complex numbers; if it is positive the roots are real.

73. Exercises.

Solve the following quadratic equations by factoring and verify the results:

1. $6x^2 = 7 + x$.
2. $21 + 8x^2 = 26x$.
3. $26x - 21 + 11x^2 = 0$.
4. $10x^2 + 26x = 5x^2 - 24$.
5. $35b^2 = 9x^2 + 6bx$.
6. $96x^2 = 4x + 15$.

Solve the following quadratics by completing the square and verify the results:

7. $5x^2 + 14x = 55$.
8. $3x^2 + 121 = 44x$.
9. $x + 22 = 6x^2$.
10. $\dfrac{5x - 1}{x + 1} = \dfrac{3x}{2}$.
11. $\dfrac{x + 3}{2x - 7} = \dfrac{2x - 1}{x - 3}$.
12. $\dfrac{3x + 1}{x + 8} + \dfrac{x - 8}{3x - 1} = \dfrac{17}{12}$.

Solve the following by formula and verify the results:

13. $3x^2 = 15 - 4x$.
14. $2x^2 + 7x = 15$.
15. $5x^2 + 4 + 21x = 0$.
16. $35 + 9x = 2x^2$.
17. $x(3x - 1) = 1$.
18. $5(x^2 + 2) = 17x$.

Chapter 14

CUBIC AND QUARTIC EQUATIONS

74. The Complete Cubic Equation. The complete cubic equation with one unknown quantity is one which contains a constant term and the first, second and third powers of the unknown but no higher power. When all terms are transposed to the left member and like terms combined, the equation may then be divided by the coefficient of the cube of the unknown and so reduced to the form

$$x^3 + ax^2 + bx + c = 0. \tag{23}$$

This will be referred to as the standard form of the *complete cubic equation* in one unknown.

If in this equation a new variable u is substituted for x by putting $x = u - \frac{a}{3}$, where a is the coefficient of the x^2 term, and the resulting equation then simplified, a new equation results, which is of the form

$$u^3 + pu + q = 0. \tag{24}$$

In this equation p and q are new coefficients which are made up of combinations of the original coefficients a, b and c. It is to be noted that there is no second power term in this equation. The second power is said to have been *suppressed* and the equation (24) is called the *reduced cubic* equation.

As an example of this reduction we will suppress the square term in the complete cubic equation

$$x^3 - 9x^2 + 9x - 8 = 0, \tag{25}$$

in which $a = -9$, $b = 9$, $c = -8$. To do this we put $x = u - a/3 = u - (-9/3)$, or

$$x = u + 3.$$

The equation then becomes

$$(u + 3)^3 - 9(u + 3)^2 + 9(u + 3) - 8 = 0.$$

Cubing and squaring the binomial $u + 3$, this is

$$u^3 + 9u^2 + 27u + 27 - 9u^2 - 54u - 81 + 9u + 27 - 8 = 0,$$

and when like terms are combined there results finally,

$$u^3 - 18u - 35 = 0, \tag{26}$$

in which there is no u^2 term.

This new cubic equation with u as the unknown quantity is more easily solved than the original equation (25). When this equation has been solved for the three values of u, the three values of x which are the roots of the original cubic are known from the relation $x = u + 3$, the number 3 being added to each value of u to give the corresponding x.

Since there are three values of u and three values of x these are denoted by u_1, u_2, u_3, and x_1, x_2, x_3, or for brevity, x_j with $j = 1, 2, 3$. The small figures or letter are not exponents and have nothing to do with powers of u or x, and neither are they coefficients. They are simply numbers or "tags" to distinguish different ones of the values of u or x and are called *subscripts.*

75. Roots of the Reduced Cubic. In order to solve the complete cubic equation it is first necessary to find the roots of the reduced cubic. The solution of this equation is itself somewhat complicated and we will not give a full and detailed explanation of the solution as in the case of the quadratic equation, but simply an outline of the method and procedure. The solution is found by formula as in the case of the quadratic but the formulas are not quite so simple as the quadratic formula.

Beginning with the reduced cubic equation

$$u^3 + pu + q = 0, \tag{24}$$

the general procedure and the significance of some of the steps are as follows:

First, using p and q (with their signs) from (24), calculate the number

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2. \tag{27}$$

Next, if D is found to be a positive number ($D > 0$) find its square root (from a table or otherwise) and using this $\sqrt{D}$ together with q from (24), calculate the two numbers

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{D}}, \quad B = \sqrt[3]{-\frac{q}{2} - \sqrt{D}}. \tag{28}$$

Finally, when A and B have been calculated, the roots of (24) are

$$u_1 = A + B,$$
$$u_2 = -\tfrac{1}{2}u_1 + \tfrac{1}{2}\sqrt{3}(A - B)i,$$
$$u_3 = -\tfrac{1}{2}u_1 - \tfrac{1}{2}\sqrt{3}(A - B)i,$$

where, as usual, $i = \sqrt{-1}$.

The roots u_2 and u_3 thus found are complex numbers, and, as their respective real parts and imaginary parts are the same, they are conjugate. If $A = B$ then $A - B = 0$ and the imaginary parts of roots u_2 and u_3 are zero. In this case these two roots are the same, $u_2 = u_3 = -\frac{1}{2}u$. In order that $A - B = 0$, or $A = B$, the formulas (28) show that $\sqrt{D}$ and therefore D must be zero; that is, according to (27),

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = 0. \tag{29}$$

If it is found that $D = 0$, therefore, the three roots u of (24) are found at once.

If the number D is found by (27) to be negative ($D < 0$) then $\sqrt{D}$ is an imaginary number and the numbers A and B of (28) cannot be calculated by our present methods. It may seem that in this case the roots u of (24) are imaginary. It is shown in higher algebra, however, that they are all three real, but the methods of *trigonometry* are required to calculate them.* In this case the reduced cubic (24) and the complete cubic (23) cannot be solved by algebraic methods.

The results of the last paragraphs show that the value of the number D enables us to determine in advance the nature of the roots of the reduced cubic (24), and therefore of those of the original cubic (23). D is therefore called the *discriminant* of (24). It is to be calculated in advance for any cubic equation of the form (24) before an attempt is made to complete the solution.

As seen in the preceding article, when the three roots u of the reduced cubic have been found, the roots of the complete cubic are $x = u - (a/3)$, where a is the coefficient of x^2 in the original equation (23).

76. Solution of the Complete Cubic Equation. We give here in the form of rules a summary of the entire procedure for the solution

* *Note.* This solution is given in the author's "Trigonometry for the Practical Man," published by D. Van Nostrand Co., New York, N. Y.

of a cubic equation as discussed in the preceding articles. This is the *Tartaglia-Cardan solution* (article 58).

(1) *By clearing fractions, transposing, and dividing by the coefficient of the cube of the unknown, express the complete cubic equation in the standard form*

$$x^3 + ax^2 + bx + c = 0.$$

(2) *If a is not zero substitute in this standard form $x = u - (a/3)$ and simplify. The reduced cubic then takes the form*

$$u^3 + pu + q = 0.$$

(3) *Using the values of p and q from this equation calculate the discriminant*

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2.$$

If this gives a negative result ($D < 0$) the roots u cannot be calculated directly by algebraic methods, though they may be found by trial or other methods.

(4) *If $D = 0$ the roots of the reduced cubic are*

$$u_1 = -\sqrt[3]{4q}, \quad u_2 = u_3 = \sqrt[3]{\frac{q}{2}}.$$

(5) *If D is positive ($D > 0$) calculate the two numbers*

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{D}}, \quad B = \sqrt[3]{-\frac{q}{2} - \sqrt{D}}.$$

(6) *The roots of the reduced cubic are then*

$$\begin{aligned} u_1 &= A + B, \\ u_2 &= -\tfrac{1}{2}u_1 + \tfrac{1}{2}\sqrt{3}(A - B)i, \\ u_3 &= -\tfrac{1}{2}u_1 - \tfrac{1}{2}\sqrt{3}(A - B)i. \end{aligned}$$

(7) *When the roots u are found in* (4) *or* (6) *the roots of the original complete cubic equation of* (1) *are*

$$x_1 = u_1 - \tfrac{1}{3}a, \quad x_2 = u_2 - \tfrac{1}{3}a, \quad x_3 = u_3 - \tfrac{1}{3}a.$$

To illustrate this method we give here the solution of the equation (25) already considered in article 74.

(1) The equation in standard form is

$$x^3 - 9x^2 + 9x - 8 = 0. \tag{25}$$

(2) In this we substitute $x = u - (a/3) = u - (-9/3) = u + 3$ to

remove the second power term. This reduction has already been carried out in article 74, and the reduced cubic is equation (26),

$$u^3 - 18u - 35 = 0. \tag{26}$$

(3) In this equation $p = -18$, $q = -35$. The discriminant is therefore

$$D = (-\tfrac{18}{3})^3 + (-\tfrac{35}{2})^2 = (-6)^3 + (\tfrac{35}{2})^2 = -216 + \tfrac{1225}{4} = +\tfrac{361}{4} > 0.$$

(5) Using $D = \frac{361}{4}$, $\sqrt{D} = \sqrt{\frac{361}{4}} = \frac{19}{2}$, $q = -35$,

$$A = \sqrt[3]{-(-\tfrac{35}{2}) + \tfrac{19}{2}} = \sqrt[3]{\tfrac{54}{2}} = \sqrt[3]{27} = 3,$$

$$B = \sqrt{-(-\tfrac{35}{2}) - \tfrac{19}{2}} = \sqrt[3]{\tfrac{16}{2}} = \sqrt[3]{8} = 2.$$

(6) The roots of the reduced cubic (26) are

$$u_1 = 3 + 2 = 5,$$

$$u_2 = -\frac{1}{2}(5) + \frac{1}{2}\sqrt{3}(3 - 2)i = -\frac{5}{2} + \frac{\sqrt{3}}{2}i,$$

$$u_3 = -\frac{1}{2}(5) - \frac{1}{2}\sqrt{3}(3 - 2)i = -\frac{5}{2} - \frac{\sqrt{3}}{2}i.$$

(7) The roots of the cubic equation (25) are then

$$x_1 = u_1 + 3 = 8,$$

$$x_2 = u_2 + 3 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i = \tfrac{1}{2}(1 + i\sqrt{3}),$$

$$x_3 = u_3 + 3 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}i = \tfrac{1}{2}(1 - i\sqrt{3}).$$

The cubic equation $x^3 - 9x^2 + 9x - 8 = 0$, therefore, has one real root, $x = 8$, and two conjugate complex roots, $x = \frac{1}{2}(1 + i\sqrt{3})$ and $x = \frac{1}{2}(1 - i\sqrt{3})$.

By the method just illustrated, any cubic equation can be solved when the discriminant is not negative. If the equation is a complete cubic, the entire procedure is necessary. If, however, the equation does not contain the square of the unknown, the steps (2) and (7) are not necessary.

In any case, if the coefficient of the cube of the unknown is different from 1, the equation must first be divided by that coefficient in order to put the equation in the form (23).

77. The Complete Quartic Equation. The standard form of the *complete quartic equation* is

$$x^4 + ax^3 + bx^2 + cx + d = 0, \tag{30}$$

and by clearing fractions, transposing and dividing the equation by the coefficient of the fourth power (if different from 1) any complete quartic can be put into this form.

The complete quartic equation, like the complete cubic, may be put into the reduced form by suppressing the term containing the third power of the unknown. This is accomplished by substituting, in (30), $x = y - (a/4)$, where a is the coefficient of the cube of the unknown, and the reduced quartic is of the form

$$y^4 + py^2 + qy + r = 0, \tag{31}$$

in which there is no term containing the third power of the new unknown y. The new coefficients p, q, and r are made up of combinations of the original coefficients of equation (30).

As an example, we will suppress the cube term in the complete quartic equation

$$x^4 + 8x^3 + 12x^2 - 11x + 2 = 0. \tag{32}$$

Here $a = 8$ and we put $x = y - (8/4) = y - 2$. This gives

$$(y - 2)^4 + 8(y - 2)^3 + 12(y - 2)^2 - 11(y - 2) + 2 = 0.$$

Raising the binomial $(y - 2)$ to the indicated powers and multiplying by the coefficients, the left member of the equation becomes

$$\begin{array}{rrrrr}
y^4 & - 8y^3 & + 24y^2 & - 32y & + 16 \\
 & + 8y^3 & - 48y^2 & + 96y & - 64 \\
 & & + 12y^2 & - 48y & + 48 \\
 & & & - 11y & + 22 \\
 & & & & + 2 \\
\hline
y^4 & & - 12y^2 & + 5y & + 24,
\end{array}$$

or, adding, (the last line above)

and the reduced quartic is

$$y^4 - 12y^2 + 5y + 24 = 0. \tag{33}$$

The solution of the reduced quartic, while it is itself a somewhat complicated affair, is much simpler than any direct solution of the complete quartic could be, even if any direct solution of the complete quartic were known.

When the four roots y_1, y_2, y_3, y_4 of the reduced quartic have been found, the four roots of the original complete quartic are then given by

$$x_j = y_j - \tfrac{1}{4}a, \quad \text{where} \quad j = 1, 2, 3, 4.$$

In the next article is given an outline of the method of solving quartic equations.

78. Solution of the Complete Quartic Equation. The method of solving the *complete quartic equation* will be given without discussion in the form of a set of rules similar to those given in article 76 for the cubic equation. This is the *Ferrari Solution* (article 58).

(1) *Express the complete quartic in the standard form*

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

(2) *In this equation substitute* $x = y - \frac{1}{4}a$ *and reduce, and it becomes*

$$y^4 + py^2 + qy + r = 0.$$

(3) *Using the coefficients of this reduced equation, write the cubic equation*

$$z^3 - pz^2 - 4rz + (4pr - q^2) = 0.$$

(4) *Find one root* z_1 *of this complete cubic by the method of article* 76, *or by trial.*

(5) *Using this value of* z_1 *and the coefficients of the reduced quartic, calculate the numbers*

$$k = \sqrt{z_1 - p}, \quad l = \frac{1}{2}\left(z_1 + \frac{q}{k}\right), \quad m = \frac{1}{2}\left(z_1 - \frac{q}{k}\right).$$

(6) *Using these values of* k, l *and* m, *write out the two quadratic equations*

$$y^2 - ky + l = 0, \quad y^2 + ky + m = 0.$$

(7) *Solve these two quadratic equations. The four values of* y *obtained are the roots of the reduced quartic equation.*

(8) *The four* x *roots of the original complete quartic are then*

$$x_j = y_j - \tfrac{1}{4}a, \quad \text{where} \quad j = 1, 2, 3, 4.$$

Quartic equations are not of frequent occurrence in the usual applications of mathematics but the rule is given as a matter of interest and for possible reference. As an illustration of its use we give here the solution of the quartic equation (32) above.

(1) The complete quartic equation in standard form is

$$x^4 + 8x^3 + 12x^2 - 11x + 2 = 0. \tag{32}$$

(2) In this equation we substitute $x = y - (a/4) = y - (8/4) = y - 2$ to remove the third power term. The reduction has already been carried out above, and the reduced quartic equation is (33),

$$y^4 - 12y^2 + 5y + 24 = 0. \tag{33}$$

(3) In (33) the coefficients are $p = -12$, $q = 5$, $r = 24$. Therefore $4r = 96$, $4pr = -1152$, $(4pr - q^2) = -1177$; and the "auxiliary cubic" equation is

$$z^3 + 12z^2 - 96z - 1177 = 0.$$

(4) By the method of article 76, or by trial, one real root of this equation is found to be $z_1 = -11$.

(5) Using $z_1 = -11$, and $p = -12$, $q = 5$ from (33), we calculate

$$k = \sqrt{-11 + 12} = \sqrt{1} = 1,$$

$$l = \frac{1}{2}\left(-11 + \frac{5}{1}\right) = \frac{-6}{2} = -3,$$

$$m = \frac{1}{2}\left(-11 - \frac{5}{1}\right) = \frac{-16}{2} = -8.$$

(6) The "equivalent quadratics" are, therefore,

$$y^2 - y - 3 = 0, \quad y^2 + y - 8 = 0.$$

(7) Solving these two quadratics (article 71),

$$y = \frac{1 \pm \sqrt{13}}{2} \quad \text{and} \quad y = \frac{-1 \pm \sqrt{33}}{2}.$$

These four values of y are the four roots of the reduced quartic (33).

(8) The four roots of the complete quartic equation (32) are

$$x = y - 2 = \frac{-3 \pm \sqrt{13}}{2} \text{ and } \frac{-5 \pm \sqrt{33}}{2}.$$

If these are desired in decimal form, they are approximately

$$x_1 = 0.303, \quad x_2 = -3.303, \quad x_3 = 0.372, \quad x_4 = -5.372.$$

79. Remarks. Equations of Higher Degree. It has been found in article 67 that an equation of the second degree may be solved by factoring, if when all the terms are transposed to one side of the equality sign the terms can then be factored as the product of two binomials, each of these factors then being set equal to zero, since their product is zero.

Similarly an equation of the third degree may be solved by factoring if, when all the terms are on one side the terms can then be factored as the product of three binomials, each of these factors then being set equal to zero, since their product is zero. In the same manner an equation of the fourth degree may be solved if with all the terms on one side these terms can be factored as the product of four binomial factors, each of these factors being set equal to zero. If the factors in either of these cases are easily determined by simple inspection or trial after arranging the terms, this method is the simplest and most direct of all. When the roots are not whole numbers or simple fractions, however, factorization in this simple manner is not possible.

The solutions of the equations of the second, third and fourth degrees given in articles 70, 76 and 78 involve only "algebraic" operations: addition, subtraction, multiplication, division, root extraction, and raising numbers to powers, and each of these only a definite finite number of times in each solution. Such solutions are called *algebraic solutions*, and the results (roots) are exact, requiring no approximation in their evaluation. (It may be that an indicated root is irrational in some cases, but the solution is complete and exact in that the final operation is indicated and written out.)

It has been proved that equations of degree higher than the fourth do not have such algebraic solutions. That is, equations of the fifth, sixth and higher degrees cannot be solved by the methods of ordinary algebra. Equations of the fifth degree may be solved by the methods developed in a branch of higher mathematics known as the theory of *elliptic functions* (see article 58). All such equations may be solved by factoring as described above for the cubic and quartic, *if they can be factored.*

The methods of Horner and Graeffe mentioned in article 58 may be used to find approximate values, expressed as decimals to any desired number of decimal places, of the roots of equations of any degree, but when the degree is high or when the equation is complete the calculations become very tedious, even when the slide rule or logarithms are used.

An extension of the method of plotting graphs of formulas and equations studied in the volume of this series on Arithmetic is to be extended in the volume on Calculus, and there certain other approximate methods of solving equations of higher degree will be developed on the basis of these graphical methods.

In recent years machines have been developed which will determine the roots of equations. These machines have been found very useful in industry, especially in engineering work. They are also used extensively in many branches of research.

80. Exercises.

Solve the following equations by the method of articles 76 and 78:

1. $x^3 - 9x - 28 = 0.$
2. $x^3 + 3x^2 + 9x - 1 = 0.$
3. $4x^3 - 7x - 6 = 0.$
4. $3x^3 - 9x^2 + 14x + 7 = 0.$
5. $x^3 - 3x - 4 = 0.$
6. $x^3 - 27x = 54.$
7. $x^3 - 8x^2 - 19x = 12.$
8. $x^3 + x^2 - 33x + 63 = 0.$
9. $x^3 - 3x^2 + 4 = 0.$
10. $x^2 - 31x = 30.$
11. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0.$
12. $4x^4 - 31x^2 + 21x + 18 = 0.$

Chapter 15

SOLUTION OF PROBLEMS BY MEANS OF EQUATIONS

81. Introduction. In the Introduction and earlier chapters of this book it has been emphasized that algebraic symbols are the "alphabet" of the mathematical language or shorthand and that the main business of algebra is to study the methods and principles of this language, the rules of the grammar of mathematics, so to speak. No problems, such as those which are met in arithmetic, have been solved and the technical application of the principles has been left for other branches of mathematics.

It was stated in the beginning, however, and has been seen in some of the chapters of Parts I and II, that in one respect algebra is a kind of extended and generalized arithmetic; and in the historical sketch in the first chapter on equations (article 58) it was seen that algebra had its origin in the study of equations as methods of solving problems.

It has been seen, also, that an equation is a statement of a condition which a certain number, represented by the "unknown quantity" in the equation, must satisfy. If, therefore, a certain condition or set of conditions involving the unknown answer to a problem can be expressed in terms of words and numbers in such a way that the statement can be translated into the mathematical shorthand, the condition can be expressed as an algebraic equation or set of simultaneous equations, and the desired result or unknown quantity found by simply solving the equation and translating the result back into the language of the original statement.

In this sense, and from this point of view, one very important object of algebra is the solution of problems by means of equations. The present chapter is, therefore, devoted to this object.

82. Algebraic Statement and Solution of Problems. In expressing the conditions of a problem by means of equations, the problem must be analyzed and the separate conditions distinguished. The known numbers and quantities involved are then to be expressed in

the ordinary way of arithmetic and any unknown but essential numbers or quantities represented by algebraic symbols. The set of relations existing among these known and unknown numbers which go to make up or specify a certain condition may then be expressed by means of the algebraic signs for addition, subtraction, powers, roots, equality, etc., and the resulting form of statement of the condition will be an algebraic equation. An equation so obtained is said to be *set up* from the problem. The solution or solutions of this equation, or associated set of equations, will then give the value or values of the desired quantities.

As an example of such a statement and solution of a problem, let us consider the simple problem given in article 2 as an illustration of arithmetical operations with algebraic symbols as numbers.

The problem stated in words is: *A boy makes a purchase which takes half the money he has. He then earns twice as much as he had at first, and finds that he has five dollars. How much did he have at first?*

Here the amount he had at first is several times referred to: half of it is said to be spent, twice the amount is said to be earned, but the amount itself is not known, although it is the essential quantity of the entire problem. The symbol x, say, is therefore assigned to represent it. That is, x is the number of dollars which the boy had at first.

It is now easy to represent symbolically the various transactions in which this amount of money, or number of dollars, is involved. Thus, the spending or removal of half of it is represented by $-\frac{1}{2}x$, and the earning or addition of twice the amount is represented by $+2x$. The final result of the several transactions is that the boy has five dollars. This means that the operations of subtracting $\frac{1}{2}x$ from x, followed by the addition of $2x$, produces, is equivalent to, or *equals* 5. In symbols, therefore, the statement of this *condition* is

$$x - \tfrac{1}{2}x + 2x = 5,$$

and this expression is a familiar simple equation with one unknown. As soon as the value of the unknown x is found and translated into the original terms, the problem is solved. To find the value of x, that is, to solve the given problem, it is only necessary to solve this simple equation. (It is this origin of equations in problems which gives us the term *to solve* to represent the operation of finding the value of the unknown in an equation.)

By the familiar method of combining like terms and dividing the

equation by the coefficient of the unknown, we find at once from the equation that $x = 2$, and, when this result is translated back into the original language of the problem, we know that the boy had two dollars to begin with.

This simple problem has been discussed at length for the purpose of showing how the statement of the conditions of problems give rise to equations, and how the algebraic solutions of the equations give the solutions of the problems. When the correspondence between the two sets of ideas involved is clear, the full power of the algebraic shorthand can be brought to bear and the process becomes greatly shortened. Thus, when the problem as stated above in italics has been analyzed, the whole procedure reduces to the following:

Let x = number of dollars boy had at first.

Then $\frac{1}{2}x$ = number of dollars spent;

$2x$ = number of dollars earned;

5 = number of dollars boy had finally.

Hence
$$x - \tfrac{1}{2}x + 2x = 5.$$

Solving,
$$x = 2.$$

Following this procedure, let us solve the following problem by means of equations: *If from three times a certain number there is subtracted twice a certain other number, the remainder is* 5, *while if twice the second number is added to four times the first the sum is* 9. *What are the two numbers?*

When this problem is analyzed, it appears that there are two unknown quantities involved, and also two conditions which they satisfy. Interpreting these two *conditions* as two *equations* and using symbols to represent the unknown numbers, the equations are at once set up as follows:

Let

x = the first number;

y = the second number.

Then the first condition is

$$\left.\begin{aligned} 3x - 2y &= 5 \\ 4x + 2y &= 9 \end{aligned}\right\}$$

and

is the second condition.

These expressions form two simultaneous equations with the two

unknowns x and y, and their solutions will give the two desired numbers. These happen to be the equations (2) of article 62, and, as there solved, the results are

$$x = 2, \quad y = \tfrac{1}{2}.$$

The two numbers of the problem stated above in italics are, therefore, 2 and $\frac{1}{2}$.

Consider next the following problem: *The difference between the perimeter and the area of a certain square, expressed in appropriate units, is* 3, *the area being the smaller. Find the length of the side of the square.*

Here only one unknown quantity is involved, the side of the square, and in terms of the side it is known that the perimeter (distance around) is four times the side and the area is the square of the side. The condition of the problem is, that when the number representing the area is subtracted from that representing the perimeter, the remainder is 3. The equation is, therefore, set up as follows:

Let $\qquad x =$ length of side of square.

Then $\qquad 4x =$ perimeter of square;

$\qquad x^2 =$ area of square;

and
$$4x - x^2 = 3.$$

This expression is a complete quadratic equation with the side x of the square as the unknown. In order, therefore, to find the side of the square, a quadratic equation is to be solved.

Rearranging the equation in the standard form, it is

$$x^2 - 4x + 3 = 0.$$

Factoring,
$$(x - 3)(x - 1) = 0.$$

Therefore,
$$x = 3, \quad x = 1.$$

It was to be expected that two values of x would be obtained as soon as it was found that the equation of the problem was a quadratic. It is now necessary to translate these results back into the language of the problem as originally stated.

Suppose we take the side of the square as 3. The perimeter is, then, 12 and the area is 9, the difference being 3, and this value, therefore, satisfies the condition of the problem. If we take the side as 1, the perimeter is 4; the area is 1, and the difference is again 3. This value of x, therefore, also satisfies the condition of the problem.

We find that there are *two* squares which fit the given description, their sides being respectively 1 and 3 units long.

The mathematical properties of the quadratic equation, therefore, automatically take care of all the circumstances of the problem and the mathematical solutions of the equation completely satisfy all the conditions of the problem. If the two solutions had been the same, it would have meant that there is only one such square.

As a final example, let us consider the following problem: *The sum of the volume of a cube and the perimeter of one face of the cube, less its total surface area, is the same as the length of its edge increased by* 18. *Find the edge of the cube.*

Here there is only one unknown, the edge of the cube, and only one condition is stated. The volume of a cube is the algebraic cube of its edge, and, since each face is a square and there are six faces, the total surface area is six times the area of one face, which is the square of the edge. The perimeter of the square face is four times the edge. Proceeding with the setting up of the equation, therefore, we first

let $x =$ length of edge of cube.

Then $x^3 =$ volume of cube;

$4x =$ perimeter of face;

$x^2 =$ area of face;

$6x^2 =$ total surface area;

$x + 18 =$ edge increased by 18.

$$\therefore \quad (x^3 + 4x) - 6x^2 = x + 18$$

is the algebraic statement of the condition of the problem.

The problem, therefore, leads to a cubic equation, and when the terms are transposed, combined and rearranged, the equation becomes

$$x^3 - 6x^2 + 3x - 18 = 0,$$

which is in the standard form of the complete cubic equation. The solution of this cubic will give the length of the edge of the cube.

The solution is carried out as in article 76. Substituting $x = u - (-6/3) = u + 2$ to suppress the square term, the reduced cubic is found to be

$$u^3 - 9u - 28 = 0.$$

The coefficients of this reduced cubic are $p = -9$ and $q = -28$. Proceeding as in article 76, therefore,

$$D = \left(\frac{-9}{3}\right)^3 + \left(\frac{-28}{2}\right)^2 = (-3)^3 + (14)^2 = -27 + 196 = +169 > 0.$$

$$\therefore \quad \sqrt{D} = \sqrt{169} = 13.$$

Then

$$A = \sqrt[3]{-\left(\frac{-28}{2}\right) + 13} = \sqrt[3]{14 + 13} = \sqrt[3]{27} = 3,$$

$$B = \sqrt[3]{-\left(\frac{-28}{2}\right) - 13} = \sqrt[3]{14 - 13} = \sqrt[3]{1} = 1;$$

and

$$u_1 = A + B = 4,$$

$$u_2 = -\frac{4}{2} + \frac{\sqrt{3}}{2}(3 - 1)i = -2 + i\sqrt{3},$$

$$u_3 = -\frac{4}{2} - \frac{\sqrt{3}}{2}(3 - 1)i = -2 - i\sqrt{3},$$

are the roots of the reduced cubic.

The roots of the original equation are, therefore (since we put $x = u + 2$),

$$x_1 = 4 + 2 = 6,$$

$$x_2 = (-2 + i\sqrt{3}) + 2 = +i\sqrt{3},$$

$$x_3 = (-2 - i\sqrt{3}) + 2 = -i\sqrt{3}.$$

The values given by the cubic equation for the edge of the cube are, therefore, 6, $i\sqrt{3}$ and $-i\sqrt{3}$. Since the last two are imaginary numbers, no actual cube exists which has these sides. Therefore, there is only one cube which satisfies the condition of the problem and its edge is 6 units in length.

The volume of this cube is $6^3 = 216$ and the perimeter of a face is $4 \times 6 = 24$. The area of a face is $6^2 = 36$ and the total surface area is $6 \times 36 = 216$. The sum of the volume and face perimeter, less total surface area is, therefore,

$$(216 + 24) - 216 = 24,$$

and this is the same as the edge increased by 18, $6 + 18 = 24$.

The mathematical properties of the cubic equation therefore not only determine the edge of the cube to fit the condition specified in the problem, but also tell us that there is only one such cube, since two of the mathematical solutions are imaginary.

The equation again automatically takes care of all the circumstances if we can but *state the conditions in algebraic symbols* and then *solve the equation.* The first part of our task is the part which requires analysis and careful thought. The second only requires that we follow the rules of algebra, and with practice this can almost be done mechanically.

Problems which lead to quartic equations are not of frequent occurrence in the usual applications of mathematics and we will not give one here.

In this article we have given an illustration of the solution of problems involving each of the more usual forms of equations and have explained them somewhat fully, both in the setting up and in the solutions. In the next article, we give a number of such illustrative problem solutions to show something of the range of application of the algebraic methods. The discussions are concise, however, and the solutions of the equations are not worked out in such detail as in the case of the problems solved above. These details are left for the reader to supply.

83. Illustrative Problem Solutions.

Problem 1. Divide 60 into two parts such that three times the greater part exceeds 100 by as much as eight times the smaller is less than 200.

Solution. Here we have two numbers to find, but, since their sum is 60, one is known as soon as the other is found. There is, therefore, really only one unknown quantity. Therefore, let x be the greater number, then $60 - x$ is the smaller. Three times the greater number is $3x$ and its excess over 100 is

$$3x - 100.$$

Eight times the smaller number is $8(60 - x)$ and this is less than 200 by

$$200 - 8(60 - x).$$

The condition of the problem is that these two differences are the same.

The equation of the problem is, therefore,

$$3x - 100 = 200 - 8(60 - x),$$

or, $$3x - 100 = 200 - 480 + 8x.$$

Solving, $$x = 36, \text{ the greater number;}$$

and $$60 - x = 24, \text{ the smaller number.}$$

Problem 2. Divide \$47 between A, B and C so that A has \$10 more than B and B has \$8 more than C.

Solution. As soon as the amount C receives is known, the amounts that A and B receive are known. Therefore,

let $$x = \text{number dollars } C \text{ receives.}$$

Then $$x + 8 = \text{number dollars } B \text{ receives.}$$

and $$x + 18 = \text{number dollars } A \text{ receives.}$$

Since the sum of the three amounts is 47 dollars we have

$$x + (x + 8) + (x + 18) = 47,$$

or, $$x + x + 8 + x + 18 = 47.$$

Solving $$x = 7 \text{ dollars, } C \text{ receives.}$$

$$x + 8 = 15 \text{ dollars, } B \text{ receives.}$$

$$x + 18 = 25 \text{ dollars, } A \text{ receives.}$$

Problem 3. The length of a room is three feet more than the width. If the length were three feet more than it is and the width two feet less than it is, the floor area would be the same. What are the dimensions of the room?

Solution.

Let $$x = \text{width in feet.}$$

Then $$x + 3 = \text{length in feet,}$$

$$(x + 3) + 3 = \text{increased length,}$$

and $$x - 2 = \text{decreased width.}$$

Since *area* = (*width*) × (*length*), the area of the room is $x(x + 3)$. With the changed dimensions the area would be $(x - 2)[(x + 3) + 3]$, or, $(x - 2)(x + 6)$. Since the area is the same in both cases

$$x(x + 3) = (x - 2)(x + 6).$$

Carrying out the indicated multiplication, this is

$$x^2 + 3x = x^2 + 4x - 12.$$

At first glance this seems to be a quadratic equation but when the terms are transposed it becomes

$$x^2 + 3x - x^2 - 4x = -12.$$

Combining,

$$-x = -12.$$

Dividing by -1,

$$x = 12 \text{ ft., width.}$$

$$x + 3 = 15 \text{ ft., length.}$$

Problem 4. If 3 pounds of butter and 5 pounds of cheese together cost $6.61 and 4 pounds of butter and 3 pounds of cheese at the same price cost $5.55, find the price per pound of each.

Solution. Here, we have two unknowns, and, as there is no simple connection between them, we must represent them by different symbols. Therefore,

Let x = the price of butter per pound in cents

and y = the price of cheese per pound in cents.

Then, $3x$ = the price of 3 pounds of butter in cents

and, $5y$ = the price of pounds of cheese in cents.

Therefore,

$$3x + 5y = 661 \tag{i}$$

Similarly, 4 pounds of butter and 3 pounds of cheese cost, together, $4x + 3y$ and this is given as $5.55, or 555 cents. Hence,

$$4x + 3y = 555 \tag{ii}$$

The equations (i) and (ii) state the conditions of the problem and are two simultaneous simple equations with the two unknowns x and y. The solutions of these equations give the solution of the problem.

In order to solve (i) and (ii) by the subtraction method, multiply (i) by 4 and (ii) by 3. This gives

$$12x + 20y = 2644$$

$$12x + 9y = 1665$$

Subtracting,

$$11y = 979$$
$$y = 89 \text{ cents, price of cheese.}$$

Substituting this value of y in equation (i),

$$3x + 445 = 661$$
$$3x = 216$$
$$x = 72 \text{ cents, price of butter.}$$

Problem 5. A number of oranges at the rate of 3 for 25 cents and apples at 70 cents a dozen, together, cost \$20.90. Five times as many oranges and one fourth as many apples at the same rate would have cost \$64.60. How many of each were bought?

Solution. Let x = the number of oranges bought

and y = the number of apples bought.

Since 3 oranges cost 25 cents, each orange cost $\frac{25}{3}$ cents, and x oranges cost $\frac{25x}{3}$ cents. Similarly, the apples at 70 cents per dozen, cost $\frac{70}{12} = \frac{35}{6}$ cents each and y apples cost $\frac{35y}{6}$. The total cost is \$20.90 or 2090 cents. Therefore,

$$\frac{25x}{3} + \frac{35y}{6} = 2090 \qquad \text{(i)}$$

If 5 times as many oranges had been bought, the cost of the oranges would have been 5 times as great, that is, $5\left(\frac{25x}{3}\right)$ or $\frac{125x}{3}$. Similarly, one fourth as many apples would cost $\frac{1}{4}\left(\frac{35y}{6}\right)$ or $\frac{35y}{24}$ cents. As the total in this case would be \$64.60, or 6460 cents, we have

$$\frac{125x}{3} + \frac{35y}{24} = 6460 \qquad \text{(ii)}$$

Equations (i) and (ii) are two simultaneous equations to be solved for x and y, the number of oranges and apples, respectively. Clearing (i) and (ii) of fractions, they become, respectively,

$$50x + 35y = 12540 \qquad \text{(iii)}$$
$$1000x + 35y = 155040 \qquad \text{(iv)}$$

Subtract (iii) from (iv) to obtain

$$950x = 142500$$

$$x = 150 \text{ oranges.}$$

Divide equation (iii) by 5. This gives

$$10x + 7y = 2508.$$

In this, substitute

$$x = 150$$

$$1500 + 7y = 2508$$

$$7y = 1008$$

$$y = 144 \text{ apples.}$$

Therefore, 12 dozen apples and $12\frac{1}{2}$ dozen oranges were bought.

Problem 6. A train travels 300 miles at a uniform speed. If the speed were five miles an hour less the trip would take two hours more time. Find the speed.

Solution. Let x be the speed in miles per hour. The time taken by the trip is the distance divided by the speed, or $\frac{300}{x}\cdot$ If the speed were 5 miles an hour less it would be $x - 5$ and the time required for the trip would be $\frac{300}{x-5}\cdot$ This time is two hours more than the actual time. The condition of the problem is therefore,

$$\frac{300}{x-5} = \frac{300}{x} + 2. \qquad \text{(i)}$$

Clearing fractions,

$$300x = 300(x - 5) + 2x(x - 5),$$

or

$$300x = 300x - 1500 + 2x^2 - 10x.$$

Transposing and combining terms, this becomes

$$-2x^2 + 10x + 1500 = 0.$$

Divide by -2,

$$x^2 - 5x - 750 = 0.$$

This is a quadratic equation to be solved for x, and the trinomial in the left member is factorable. Factoring it,

$$(x - 30)(x + 25) = 0,$$

$$x = 30, \quad x = -25.$$

The speed is, therefore, 30 miles per hour, the negative value being inadmissible because it does not satisfy the physical conditions of the problem, even though it does satisfy the equation mathematically.

The solution -25 miles per hour has a definite significance, however, even though it does not satisfy the physical conditions of the problem as stated. Suppose that in the original equation (i) $-x$ is put in the place of x; this gives

$$\frac{300}{-x - 5} = \frac{300}{-x} + 2,$$

or

$$-\left(\frac{300}{x + 5}\right) = -\left(\frac{300}{x} - 2\right),$$

or,

$$\frac{300}{x + 5} = \frac{300}{x} - 2. \tag{ii}$$

Comparing equation (ii) with (i) and with the statement of the original problem, it is seen that (ii) states that if the speed had been five miles an hour *more* the time would have been two hours *less*.

The solutions of equation (ii) are $x = 25$ or -30. The negative solution of the equation (i), therefore, does not apply to the problem as stated but to the *reciprocal* problem.

Problem 7. A tank can be filled in $33\frac{1}{3}$ minutes by two pipes running together. Running separately the smaller takes 15 minutes longer to fill the tank than the larger. How long does it take each to fill the tank?

Solution. Let x represent the number of minutes taken by the smaller pipe to fill the tank. Then, $x - 15$ is the number of minutes taken by the larger.

In one minute, therefore, the smaller pipe fills $\frac{1}{x}$th part of the tank and the larger $\frac{1}{(x - 15)}$th, and together they fill the

$$\left(\frac{1}{x} + \frac{1}{x - 15}\right)$$

part in one minute. But, since together they fill it in $33\frac{1}{3}$ minutes, in one minute they fill $\frac{1}{(33\frac{1}{3})}$ of it, or $\frac{3}{100}$. Therefore,

$$\frac{1}{x} + \frac{1}{x-15} = \frac{3}{100}.$$

Clearing fractions,

$$100x + 100(x - 15) = 3x(x - 15),$$

or,

$$100x + 100x - 1500 = 3x^2 - 45x.$$

Transpose and combine,

$$3x^2 - 245x + 1500 = 0.$$

This is a quadratic equation with x as unknown and the trinomial first member is factorable. Factoring it,

$$(x - 75)(3x - 20) = 0,$$

$$x = 75, \quad x = 6\tfrac{2}{3}.$$

Thus, the smaller pipe takes 75 or $6\frac{2}{3}$ minutes to fill the tank.

Since the larger pipe requires 15 minutes less it takes 60 or $-8\frac{1}{3}$ minutes. As time never flows backward, a negative time has no meaning here and the time $-8\frac{1}{3}$ here has no physical significance, even though mathematically it satisfies the conditions. Therefore, the root $6\frac{2}{3}$ of the quadratic is rejected and the time required to fill the tank is 75 minutes for the smaller pipe and 60 minutes for the larger.

Problem 8. A and B are cities 300 miles apart. Trains start from both stations at the same time, each train going toward the other city. After they meet it takes the A train nine hours to reach B and the B train four hours to reach A. Find the speed of each train.

Solution. Let x miles per hour be the rate of the train from A and y miles per hour that of the B train. Then, since *distance* = (*time*) $\times$ (*speed*), the A train travels $9x$ miles after the meeting and the B train $4y$ miles. The sum of these two distances is the distance between the stations. Therefore,

$$9x + 4y = 300 \qquad \text{(i)}$$

The distance from the meeting point to B is the distance travelled by the A train after the meeting, $9x$ miles. As *time* = (*distance*) $\div$ (*speed*), the time taken by the B train to cover this distance in reaching

the meeting point was $\frac{9x}{y}$ hours. Similarly, the time taken by the A train to reach the meeting point was $\frac{4y}{x}$ hours. As the trains started at the same time, they had been travelling for the same time when they met. Therefore,

$$\frac{4y}{x} = \frac{9x}{y}.$$

Clearing fractions

$$4y^2 = 9x^2, \tag{ii}$$

or,

$$9x^2 - 4y^2 = 0. \tag{iii}$$

Writing the equations (i) and (iii) together,

$$9x + 4y = 300 \tag{i}$$

$$9x^2 - 4y^2 = 0. \tag{iii}$$

These are simultaneous equations in two unknowns, x and y, and it is to be noted that one is a quadratic and the other a simple equation.

Simultaneous equations in which one or more are quadratics are called *simultaneous quadratic* equations. There are no definite simple rules for solving such equations but frequently one unknown can be eliminated by some simple transformation and the two equations combined into one equation with one unknown, and this equation may be simple or quadratic. In the case of simultaneous quadratic equations there are two solutions for each unknown.

In the present case one of the unknowns can be eliminated by what is known as the *method of substitution.* In this method one of the equations is solved for one of the unknowns in terms of the other. This value is then substituted in the other equation, giving an equation containing only one unknown. (This can also be used in the case of simultaneous simple equations.)

To solve the equations (i) and (iii) we shall first eliminate y and solve for x. This may be done by substitution in either of two ways. First let us solve (iii) for y and substitute in (i). Using the transposed form (ii) of equation (iii), take the square root of both sides. This gives

$$2y = \pm 3x,$$

$$y = \pm \tfrac{3}{2}x. \tag{iv}$$

Substitute this in equation (i). This gives

$$9x \pm 6x = 300.$$

Using the plus sign,

$$15x = 300.$$

$$\therefore \quad x = 20 \text{ miles per hour.}$$

Using the minus sign,

$$3x = 300.$$

$$\therefore \quad x = 100 \text{ miles per hour.}$$

Substituting these values of x in (iv) we find for y, corresponding to the plus and minus signs, respectively,

$$y = 30, \quad y = -150 \text{ miles per hour.}$$

Thus the train from A travelled at a speed of 20 or 100 miles per hour, and that from B at a speed of 30 or -150 miles. Both sets of values satisfy the equations but the 100 and -150 mile speeds do not satisfy the physical conditions of the problem. The results are, therefore, 20 and 30 miles per hour, respectively, for the A and B trains.

Making the substitution from equation (i) instead of (iii), we have from (i),

$$y = \tfrac{3}{4}(100 - 3x). \qquad \text{(v)}$$

Substituting this in equation (iii),

$$9x^2 - 4[\tfrac{3}{4}(100 - 3x)]^2 = 0,$$

or,

$$9x^2 - \tfrac{9}{4}(100 - 3x)^2 = 0.$$

Multiply by $\frac{4}{9}$,

$$4x^2 - (100 - 3x)^2 = 0.$$

Square the binomial,

$$4x^2 - 10{,}000 + 600x - 9x^2 = 0.$$

Combine,

$$-5x^2 + 600x - 10{,}000 = 0.$$

Divide by -5,

$$x^2 - 120x + 2000 = 0.$$

Factor,

$$(x - 20)(x - 100) = 0.$$

$$\therefore \quad x = 20, \quad x = 100,$$

as before, and these values in (v) give

$$y = \tfrac{3}{4}(100 - 60) = 30,$$
$$y = \tfrac{3}{4}(100 - 300) = -150.$$

Problem 9. What are the two numbers whose sum is 20 and whose product is 24 times their difference?

Solution. At first glance it seems that we have here two unknowns, but, as one member is known as soon as the other is given, there is really but one unknown. So, let x be the larger of the two numbers. Then, $20 - x$ is the smaller and their difference is $x - (20 - x)$, or $2x - 20$. The product of the two numbers is $x(20 - x)$ and this is 24 times the difference just expressed. The equation of condition is, therefore,

$$x(20 - x) = 24(2x - 20),$$

or,

$$20x - x^2 = 48x - 480.$$

Transpose,

$$-x^2 - 28x + 480 = 0.$$

Divide by -1,

$$x^2 + 28x - 480 = 0.$$

Solve,

$$x = 12, \quad \text{or} \quad -40.$$
$$20 - x = 8, \quad \text{or} \quad 60.$$

There are, therefore, two pairs of numbers which satisfy the given conditions. They are 12 and 8, or 60 and -40.

Problem 10. A lawn 50 feet long and 34 feet wide is to have a path around it of uniform width and area 540 square feet. How wide must the path be?

Solution. Let x be the width of the path in feet. The outer dimensions of the entire lawn including the path are then $50 + 2x$ and $34 + 2x$ feet, and the area of the entire lawn is

$$(50 + 2x)(34 + 2x)$$

square feet. The area of the inner lawn alone is $50 \times 34 = 1700$ square feet. The difference between these two areas is the area of the path. Therefore,

$$(50 + 2x)(34 + 2x) - 1700 = 540,$$

or

$$1700 + 168x + 4x^2 - 1700 = 540.$$

Transpose,

$$4x^2 + 168x - 540 = 0.$$

Divide by 4,

$$x^2 + 42x - 135 = 0.$$

Solve,

$$x = 3, \quad x = -45.$$

Since the negative value does not satisfy the physical conditions of the problem, the width of the path is 3 feet.

Problem 11. The perimeter of a rectangular field is 500 yards and its area is 14,400 square yards. Find its dimensions.

Solution. Let x be the length in yards. Then, since (*length*) $\times$ (*width*) $= 14{,}400$, the width is $14{,}400 \div$ (*length*) or $\frac{14{,}400}{x}$. The perimeter equals twice the length plus twice the width, and is 500 yards. Therefore,

$$2x + 2\left(\frac{14{,}400}{x}\right) = 500.$$

Clear fractions,

$$2x^2 + 28{,}800 = 500x.$$

Transpose and divide by 2,

$$x^2 - 250x + 14{,}400 = 0.$$

Solve,

$$x = 90, \quad \text{or} \quad 160 \text{ yards, length.}$$

Hence,

$$\frac{14{,}400}{x} = 160, \quad \text{or} \quad 90 \text{ yards, width.}$$

The dimensions of the field are therefore 90×160 yards.

84. Problems for Solution.

Solve the following problems by means of the appropriate equations:

1. The sum of two numbers is 29 and one is greater than the other by 5. Find the numbers.
2. Find the number which multiplied by 4 exceeds 40 by as much as 40 exceeds the number.
3. The difference between the squares of two consecutive numbers is 35. What are they?
4. An estate is divided among three persons, A, B and C. A's share is

three times that of B and B's share is twice that of C. If A receives \$9000 more than C, how much does each receive?

5. A father is four times as old as his second son and in twenty years will be twice as old as his first son, who is five years older than the second. How old is each?

6. One third the sum of two numbers is fourteen and half the difference is four. Find the numbers.

7. In a bag containing black and white balls half the number of white equals a third the number of black balls, and twice the whole number exceeds three times the number of blacks by four. How many balls are there in the bag?

8. A boy was sent for \$1.20 worth of eggs. On his way home he broke five and the actual cost was thereby twenty-four cents more per dozen than the original purchase price. How many eggs did he buy?

9. By lowering the price of apples 10 cents per dozen, a dealer sells for \$8.40, 24 apples more than before. What was the original price per dozen?

10. A broker bought a certain number of shares of stock for \$1875. He reserved fifteen shares and sold the remainder for \$1740, gaining four dollars a share on those he sold. How many shares did he buy?

11. In the center of a square garden is a square lawn; outside this is a gravel walk four feet wide, and then a flower border six feet wide. The flower border and the center lawn together have an area of 721 square feet. Find the area of the center lawn.

12. Two rectangles contain the same area, 480 square yards each. The difference of their lengths is ten yards and of their widths four yards. Find the dimensions of each.

13. A number consists of three digits, the right-hand one being zero. If the left-hand and middle digits be interchanged the number is diminished by 180; if the left-hand digit be halved and the middle and right digits interchanged, the number is diminished by 454. Find the number.

14. A man walks 35 miles, partly at the rate of 4 miles per hour and partly at 5; if he had walked at 5 miles an hour when he walked at 4, and vice versa, he would have covered 2 miles more in the same time. Find the total time he was walking.

15. A chemist with two mixtures of a chemical and water, of 20% and 40% concentrations, wishes to mix the two so as to obtain 10 ounces of 32% concentration. How much of each must he use?

16. A lawn of 50 by 34 feet has a walk of uniform width around it, the area of the walk being 540 square feet. Find the width of the walk.

17. The telegraph poles along a railroad are equally spaced. If the spacing is increased by 22 feet there will be 8 poles less per mile. How many are there in a mile and what is the spacing?

18. A baseball team has won 5 of 13 games played, and a second team has won 9 out of 15 games. How many games straight must the first team win from the second in order that each may have the same percentage record?

19. A man's income is \$5000 per year. After deducting a percentage for a state tax, and then deducting a local tax on the remainder, less by 1% than

the state tax rate, his net income is \$4,656. Find the percentage rate of the state tax.

20. A man started to walk 3 miles, intending to arrive at his destination at a certain time. After walking a mile he was detained 10 minutes, and as a result had to walk a mile an hour faster to reach his destination on time. What was his original speed?

Part 4

LOGARITHMS

Chapter 16

PRINCIPLES OF LOGARITHMS

85. Definitions. We have seen how the principles of exponents, studied in Chapters 6 and 7, enable us to simplify algebraic calculations involving multiplication, division, powers and roots. We shall see how these principles will enable us to shorten these operations with ordinary arithmetical numbers and to find any power or any root of any number by a simple multiplication or division.

In studying such expressions as $4^2 = 16$ or in general $a^n = N$, where a and n may be any real numbers, the exponent n was considered with reference to the number a, being called the "exponent of a." Here we shall consider n with reference to the number N instead and call n the "*logarithm* of N." When n is thought of as the exponent of a, the number N is the value of the power of a indicated by n. Thus in the above equations, N is the nth power of a, 16 is the 2nd power of 4, etc. When n is thought of as the logarithm of N, the number a is called the *base* of the logarithm. Thus, in the expression $a^n = N$, n is the logarithm of the number N *to the base* a. Similarly, 2 is the logarithm of 16 to the base 4.

This relation is concisely expressed in the following

DEFINITION: *The logarithm of a number to any base is the exponent of the power to which the base must be raised in order to equal the given number.*

The expression "logarithm of N to base a" is written $\log_a N$. Expressed in formulas the entire definition is contained in the following: if

$$\left.\begin{aligned} a^n &= N \\ \text{then} \qquad n &= \log_a N. \end{aligned}\right\} \tag{1}$$

Thus, $5^3 = 125$ and $3 = \log_5 125$, $10^2 = 100$ and $\log_{10} 100 = 2$, $10^3 = 1000$ and $\log_{10} 1000 = 3$, etc.

Now, as seen in Chapter 7, a root of a number may be expressed as a fractional power, thus $\sqrt{x} = x^{1/2}$, $\sqrt[3]{8} = 8^{1/3} = 2$, and so $\log_8 2 = \frac{1}{3}$. Fractional logarithms are generally written as decimal numbers. Thus we would write $\log_8 2 = .333+$. Similarly $\sqrt{4^3} =$

$4^{3/2} = 4^{1.5} = 8$, and, therefore, $\log_4 8 = 1.5$; $10^{2.574} = 375$ and $\log_{10} 375 = 2.574$; etc.

When a logarithm is fractional, as in the last few examples, the decimal part is called the *mantissa* and the integral part (to the left of the decimal point) is called the *characteristic* of the logarithm. In the examples above the characteristic of $\log_{10} 375$ is 2 and the mantissa is .574; the characteristic of $\log_4 8$ is 1 and the mantissa is .5; the characteristic of $\log_8 2$ is 0 and the mantissa is .333; etc. If the mantissa of a logarithm is not exact, it may be expressed to as many decimal places as desired.

In giving the logarithm of a number, the base must always be specified unless it is understood from the beginning that in any discussion a certain number is to be used as base for all logarithms. Any real number except 1 may be used as base, but we shall see later that in applications of logarithms only two bases are in common use. All the logarithms of numbers taken with reference to any one base are referred to as a *system* of logarithms.

In any system of logarithms, the number corresponding to any given logarithm is called an *anti-logarithm*. In the examples given above, where $\log_{10} 1000 = 3$, the number 1000 is the anti-logarithm of 3; where $\log_{10} 375 = 2.574$, 375 is the anti-logarithm of 2.574; where $\log_4 8 = 1.5$, 8 is the anti-logarithm of 1.5; etc.

86. Logarithm of a Product. Suppose that we have $\log_a A = m$ and $\log_a B = n$. An important question is: What is the logarithm of the product AB to the same base a? That is, $\log_a (AB) = ?$ The answer to the question is found by making use of the law of exponents for a product.

If

$$\log_a A = m, \quad \log_a B = n, \tag{2}$$

then by equation (1),

$$A = a^m, \qquad B = a^n. \tag{3}$$

The product is, therefore,

$$AB = a^m a^n,$$

but by the law of exponents for a product,

$$a^m a^n = a^{m+n},$$

or,

$$AB = a^{m+n}.$$

Hence, by equation (1),

$$\log_a (AB) = m + n,$$

or, by equations (2),

$$\log_a (AB) = \log_a A + \log_a B. \tag{4}$$

This fundamental formula states that

The logarithm of the product of two numbers to any base equals the sum of the logarithms of the two numbers to the same base.

As an example, $\log_2 8 = 3$, $\log_2 32 = 5$ and $\log_2 (8 \times 32) = \log_2 256 = 3 + 5 = 8$. Also $\log_{10} 100 = 2$, $\log_{10} 375 = 2.574$ and $\log_{10} (100 \times 375) = \log_{10} 37{,}500 = 2 + 2.574 = 4.574$, etc.

Similarly, for any number of factors, by repeating the above demonstration it is found that

$$\log_a (ABC) = \log_a A + \log_a B + \log_a C,$$

and so on. This gives the general rule, for any base:

The logarithm of a product is the sum of the logarithms of the factors.

87. Logarithm of a Quotient. The logarithm of a quotient is found in a manner similar to that used in the preceding article, by making use of the law of exponents for division.

According to equations (3) above, for the two numbers A and B given in equations (2),

$$\frac{A}{B} = \frac{a^m}{a^n},$$

and by the law of exponents for a quotient

$$\frac{a^m}{a^n} = a^{m-n}.$$

$$\therefore \quad \frac{A}{B} = a^{m-n}.$$

Hence, by equations (1),

$$\log_a \left(\frac{A}{B}\right) = m - n,$$

or, by (2),

$$\log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B. \tag{5}$$

In words,

The logarithm of a quotient, to any base, is the difference of the logarithms of the dividend and divisor,

(the logarithm of the divisor being subtracted).

For example, $\log_{10} 1000 = 3$, $\log_{10} 375 = 2.574$ and $\log_{10}\left(\frac{1000}{375}\right) = \log_{10} 2.67 = 3 - 2.574 = .426$.

By combining the formulas for products and quotients, we can write

$$\begin{aligned}\log_a\left(\frac{AB}{CD}\right) &= \log_a (AB) - \log_a (CD)\\ &= (\log_a A + \log_a B) - (\log_a C + \log_a D)\\ &= \log_a A + \log_a B - \log_a C - \log_a D,\end{aligned}$$

or, in multiplication logarithms are *added* and in division they are *subtracted*.

88. Logarithm of a Power. It is required to find $\log_a (A^p)$.

Let $\log_a A = n.$

Then, as before, $A = a^n$

and $A^p = (a^n)^p$

But by the law of exponents for powers,

$$(a^n)^p = a^{pn}.$$

$$\therefore \quad A^p = a^{pn}.$$

Therefore, $\log_a (A^p) = pn,$

and, since $n = \log_a A$, we have, finally,

$$\log_a (A^p) = p \cdot \log_a A. \tag{6}$$

That is,

The logarithm of any power of a number equals the logarithm of the number multiplied by the exponent of the power.

For example $\log_{10} (375^2) = 2 \cdot \log_{10} 375 = 2 \times 2.574 = 5.148$, that is, $\log_{10} 140{,}625 = 5.148$.

The proof given above for this important principle is mathematically correct but the following may be more easily understood.

By definition $A^2 = AA$, hence, $\log_a (A^2) = \log_a AA$, and, by formula (4) for the logarithm of a product, this is equal to $\log_a A + \log_a A$. That is,

$$\log_a (A^2) = \log_a (AA) = \log_a A + \log_a A = 2 \cdot \log_a A.$$

Similarly,

$$\log_a (A^3) = \log_a (AAA) = \log_a A + \log_a A + \log_a A = 3 \cdot \log_a A,$$

$$\log_a (A^4) = 4 \cdot \log_a A,$$

and, in general,

$$\log_a (A^p) = p \cdot \log_a A. \tag{6}$$

In this formula a is any base and p is any power, integral or fractional, positive or negative, to which the number A may be raised.

89. Logarithm of a Root. If, in formula (6) above, the exponent p is equal to $\frac{1}{q}$, where q is another number, the formula becomes

$$\log_a (A^{1/q}) = \frac{1}{q} (\log_a A) = \frac{(\log_a A)}{q}, \tag{7a}$$

and this does not indicate that the number A is divided by q, but that the number which is the *logarithm* of A is divided by q.

Now, a fractional power indicates a root. Therefore, $A^{1/q} = \sqrt[q]{A}$. The formula (7a) above, therefore, gives the result:

$$\log_a (\sqrt[q]{A}) = \frac{\log_a A}{q}. \tag{7}$$

In words, this states that,

The logarithm of a root of a number equals the quotient of the logarithm of the number by the index of the root.

For example,

$$\log_{10} (\sqrt{375}) = \frac{\log_{10} 375}{2} = \frac{2.574}{2} = 1.287,$$

or

$$\log_{10} 19.36 = 1.287.$$

90. Transformation of Logarithms from One Base to Another. Suppose the logarithm of a number in one system is known and it is desired to find the logarithm of the same number in some other system. This means that the logarithm of the number is taken with respect to two bases. It is sometimes important to be able to calculate one logarithm when the other is known.

Suppose the logarithm of a number N to the base a is known and it is desired to find the logarithm of N to some other base b. Let

$$x = \log_b N, \tag{8}$$

x being at present unknown. From this, by definition,

$$b^x = N,$$

and, since b^x and N are equal, they have the same logarithms, to any base. Therefore,

$$\log_a (b^x) = \log_a N.$$

But, by formula (6) $\log_a (b^x) = x \cdot \log_a b$. Therefore,

$$x \cdot \log_a b = \log_a N.$$

Substituting in this the value of x given by (8) we have,

$$\log_b N \cdot \log_a b = \log_a N,$$

and, on dividing this equation by $(\log_a b)$,

$$\log_b N = \frac{\log_a N}{\log_a b}. \tag{9}$$

That is, *to transform logarithms from any base a to another base b, divide the first logarithm by the logarithm of b to the base a.*

Formula (9) may be written

$$\log_b N = \left(\frac{1}{\log_a b}\right) \log_a N, \tag{10}$$

which means that in order to transform logarithms from base a to base b, multiply by the number $\dfrac{1}{\log_a b}$. This transforming multiplier is called the *modulus* of the system in which the base is a and is known as soon as the two bases a and b are specified.

91. Other Properties of Logarithms. The formulas or equations (1) in article 85 express the *definitions* of a logarithm and a logarithm base, while the formulas (4), (5), (6), (7) and (9) or (10) express five very important principles or *properties* of logarithms. There are a few more important properties of logarithms which we add here.

No matter what the value of a may be, we know that $a^0 = 1$. Therefore, by the definitions (1), $\log_a 1 = 0$, or for any base,

$$\log 1 = 0. \tag{11}$$

That is, *in any system the logarithm of* 1 *is zero.*

For any number whatever, by the definition of an exponent, $a^1 = a$. Therefore, $\log_a a = 1$. That is, *in any system the logarithm of the base itself is* 1.

92. Exercises.

1. Given $3^4 = 81$, what is the logarithm of 81 to the base 3?
2. Given $10^4 = 10{,}000$, what is $\log_{10} 10{,}000$?
3. Given $\log_2 8 = 3$ and $\log_2 16 = 4$, find $\log_2 128$.
4. Given $\log_{10} 260 = 1.415$ and $\log_{10} 2 = .301$, find $\log_{10} 520$.
5. Given $\log_2 512 = 9$ and $\log_2 32 = 5$, find $\log_2 16$.
6. Given $\log_{10} 732 = 2.865$ and $\log_{10} 12 = 1.079$, show that $\log_{10} 61 = 1.785$.
7. Given $\log_2 256 = 8$, find $\log_2 (256^3)$.
8. Given $\log_{10} 851 = 2.930$, find $\log_{10} (851^{2.5})$.
9. Given $\log_{10} 97.7 = 1.990$, find $\log_{10} (\sqrt[3]{97.7})$.
10. Given $\log_{10} 45 = 1.653$, show that $\log_{10} (\sqrt{45^3}) = 2.480$.

Chapter 17

SYSTEMS AND TABLES OF LOGARITHMS

93. Systems of Logarithms. As already stated, there are but two systems of logarithms in common use. These are the system in which the base is the number 10 and that in which the base is the number 2.71828+. This number is denoted by the symbol e. (In the theory of electricity where the letter e is used for *potential* the number 2.71828+ is denoted by the Greek letter "epsilon," written ε or ϵ). Logarithms to the base 10 are called *common logarithms* because of their simplicity and their use in ordinary calculations. Logarithms to the base e are called *natural logarithms* because of the manner in which they make their appearance in the higher mathematics.

The theory of natural logarithms is studied in the calculus and they are used in the formulas and calculations of physics and certain branches of engineering. For all ordinary calculations of arithmetic, however, common logarithms are used and we shall not discuss the natural logarithms in this book.* We do, however, give the formulas for converting natural logarithms into common logarithms and vice versa.

These formulas are derived from the transformation formula (10) of article 90. That formula is

$$\log_b N = \left(\frac{1}{\log_a b}\right) \log_a N, \tag{10}$$

in which N is the number whose logarithm is considered, and a and b are the two bases, a being the base to which the logarithm of N is known, and b the base to which it is desired to refer the logarithm.

In the present case, let a represent the base $e = 2.71828+$ and b the base 10. The modulus $(1/\log_a b)$ is, therefore, $1/\log_e 10$, and, by

* See the author's "Calculus for the Practical Man," published by D. Van Nostrand.

the method referred to above, it is found that $\log_e 10 = 2.3026$. Hence the modulus is $1/2.3026$ and (10) becomes

$$\log_{10} N = \left(\frac{1}{2.3026}\right) \log_e N.$$

From this we have

$$\left.\begin{aligned} \log_{10} N &= 0.4343 \log_e N \\ \log_e N &= 2.3026 \log_{10} N. \end{aligned}\right\} \qquad (12)$$

By means of these formulas, either the common or the natural logarithm of any number may be calculated from the other.

94. Common Logarithms. Using the base 10, the definition formulas (1), article 85, are

$$n = \log_{10} N, \quad N = 10^n.$$

These state that when 10 is the base of logarithms of numbers, every number is considered as some power of 10.

We write out here a few of the powers of 10.

$$\begin{aligned} 10^0 &= 1 \\ 10^1 &= 10 \\ 10^2 &= 100 \\ 10^3 &= 1{,}000 \\ 10^4 &= 10{,}000, \text{ etc.} \end{aligned}$$

This table shows that all positive integral (whole-number) powers of 10 are found simply by writing the figure 1 followed by a number of zeros or naughts equal to the exponent of 10.

Consider now the fractions 1/10, 1/100, 1/1000, etc. Expressed as decimals these are .1, .01, .001, etc. But by the law of negative exponents $\frac{1}{10} = 10^{-1}$, $\frac{1}{100} = \frac{1}{10^2} = 10^{-2}$, $\frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$, etc. Therefore we can write

$$\begin{aligned} 10^{-1} &= .1 \\ 10^{-2} &= .01 \\ 10^{-3} &= .001 \\ 10^{-4} &= .0001, \text{ etc.} \end{aligned}$$

This table shows that all negative integral powers of 10 are decimal

fractions consisting of the figure 1 *preceded* by a number of zeros equal to one less than the negative exponent of 10.

Writing the positive and negative powers of 10 in a continued series running both ways from $10^0 = 1$ we have (the dots meaning "and so on"):

$$
\begin{aligned}
\ldots\ldots &\quad \ldots \\
.00001 &= 10^{-5} \\
.0001 &= 10^{-4} \\
.001 &= 10^{-3} \\
.01 &= 10^{-2} \\
.1 &= 10^{-1} \\
1 &= 10^{0} \\
10 &= 10^{1} \\
100 &= 10^{2} \\
1{,}000 &= 10^{3} \\
10{,}000 &= 10^{4} \\
100{,}000 &= 10^{5} \\
\ldots\ldots &\quad \ldots
\end{aligned}
$$

Now, when a number is expressed as a power of 10 the exponent is the *logarithm* of the number to the base 10, that is, the common logarithm of the number. From the above table, it is seen at once that

$$\log_{10} 1 = 0$$

$$
\begin{array}{llll}
\log_{10} .1 = -1 & & \log_{10} 10 = 1 \\
\log_{10} .01 = -2 & & \log_{10} 100 = 2 \\
\log_{10} .001 = -3 & & \log_{10} 1{,}000 = 3 \\
\log_{10} .0001 = -4 & & \log_{10} 10{,}000 = 4 \\
\log_{10} .00001 = -5 & & \log_{10} 100{,}000 = 5 \\
\text{etc.} & & \text{etc.}
\end{array}
$$

From this table, the logarithms of all numbers greater than 1 are positive and the logarithms of all fractions are negative. Also, and this is very important, the common logarithms of the even tens, hundreds, thousands, etc., are 1 *less* than the number of figures in the

numbers themselves (to the left of the decimal point), while the common logarithms of the even tenths, hundredths, thousandths, etc., are 1 *more* than the number of zeros immediately to the *right* of the decimal point.

Again, it is seen at a glance that the logarithm of any number between 10 and 100 is between 1 and 2; the logarithm of any number between 100 and 1,000 is between 2 and 3; that of any number between 1,000 and 10,000 is between 3 and 4; etc.; while the logarithm of any decimal fraction between .1 and .01 is between -1 and -2; the logarithm of any between .01 and .001 is between -2 and -3; that of any between .001 and .0001 is between -3 and -4; etc. Thus $\log_{10} 93 = 1.+$, $\log_{10} 150 = 2.+$, $\log_{10} 375 = 2.574$, $\log_{10} 1265 = 3.+$, etc., while $\log_{10} .63 = -1.+$, $\log_{10} .05 = -2.+$, $\log_{10} .0743 = -2.+$, $\log_{10} .0035 = -3.+$, etc.

Summarizing the above results it is now obvious that the even tens, hundreds, thousands, etc., have integral positive logarithms, that is, the mantissas are zero, while all other numbers greater than 1 have logarithms consisting of a characteristic and a mantissa; and that the characteristic is the same for all numbers between 10 and 100, for all numbers between 100 and 1,000, between 1,000 and 10,000, etc. Similarly, even tenths, hundredths, thousandths, etc., have integral negative logarithms, i.e., zero mantissas, while other fractions have logarithms consisting of negative characteristics and positive mantissas. We write here a few more illustrative examples:

$$\begin{array}{llll}
\log_{10} .126 & = -1.+ & \log_{10} 3. & = 0.+ \\
\log_{10} .3 & = -1.+ & \log_{10} 9.0 & = 0.+ \\
\log_{10} .027 & = -2.+ & \log_{10} 5.27 & = 0.+ \\
\log_{10} .0873 & = -2.+ & \log_{10} 11.3 & = 1.+ \\
\log_{10} .00595 & = -3.+ & \log_{10} 19.36 & = 1.287 \\
\log_{10} .003 & = -3.+ & \log_{10} 487.6 & = 2.+ \\
 & \text{etc.} & \log_{10} 375.0 & = 2.574 \\
 & & \log_{10} 1{,}563 & = 3.+ \\
 & & \log_{10} 45{,}745 & = 4.+ \\
 & & \log_{10} 140{,}625 & = 5.148 \\
 & & & \text{etc.}
\end{array}$$

The three logarithms in the right-hand column given with mantissas have been found in some of the calculations of the preceding chapter.

The following laws of the characteristics of common logarithms are now obvious:

The characteristic of the logarithm of any number greater than 1 *is a positive number which is* 1 *less than the number of figures to the left of the decimal point in the number.*

The characteristic of the logarithm of any decimal fraction is a negative number which is 1 *greater than the number of zeros immediately to the right of the decimal point in the fraction.*

These two laws are of the *very greatest importance* and MUST BE LEARNED.

95. Tables of Logarithms. In the lists of examples given above the mantissas of certain logarithms are zero, but for all others they are not zero. Only those numbers which are integral powers of 10 have zero mantissas. Logarithms with zero mantissas are, therefore, comparatively few. Since the other numbers are not exact powers of 10, their logarithms are decimals which never "come out" exact. This means that the decimal mantissas may be calculated to as many places as may be desired. Thus we have $\log_{10} 100. = 2.0000000$ exactly but $\log_{10} 375$ has been seen to be 2.574, more exactly it is 2.5740313, and may be carried to any number of decimal places.

As seen above, the characteristic of the common logarithm of any number may be found at once by inspection. The mantissa, however, must be calculated by special methods. These methods are explained in books on the calculus* and will not be discussed here although a partial explanation is given in article 121 of this book. The mantissa is the same for the logarithms of all numbers consisting of the same figures. Thus $\log_{10} 375 = 2.574$, $\log_{10} 37.5 = 1.574$, $\log_{10} 3750 = 3.574$, etc., and, as shown in the columns of numbers given previously, all *mantissas* are to be *added* to their characteristics.

It is, therefore, sufficient to calculate one set of mantissas for a certain range of numbers and list these in a table, each mantissa corresponding to a certain number or group of figures. In order to determine the logarithm of *any* number, it is then only necessary to find in the table the mantissa corresponding to the figures of the number, and determine the characteristic by means of the position of

* See the author's "Calculus for the Practical Man," D. Van Nostrand Co., Princeton, N.J.

the decimal point in the number, according to the laws of the characteristics given in the preceding article. Such a table of mantissas is called a table of logarithms or a *logarithm table.*

A logarithm table may give the logarithms (mantissas) of all numbers from 10 to 100, 100 to 1,000, 1,000 to 10,000, etc., and the mantissas may be given to three, four, five, six, etc., decimal places, more exact work being possible with logarithms of more decimal places. In most branches of engineering, physics, and chemistry four-place or five-place tables are sufficient, but in surveying six- or seven-place tables are necessary and in the calculations of astronomy seven, eight or ten places are necessary and sixty-place tables have been calculated.

In the back of this book is a four-place table of the logarithms (mantissas) of all numbers from 100 to 1,000. This table and the method of using it are described in the following article.

96. How to Use the Table. The table in this book occupies two pages. On the first page are the logarithms of the numbers 100 to 549, and on the second, logarithms of numbers 550 to 1000. The first two figures of these numbers (10 to 54, 55 to 99) are in the first column at the left of each page, and the third figure (0 to 9) in a line across the top of each page.

The mantissa of the logarithm of any two-figure number is given in the second column, at the right of the number and on the line with it. Thus the mantissa of $\log_{10} 41$ is .6128 and that of $\log_{10} 75$ is .8751.

The mantissa of the logarithm of any three-figure number is on the same line with the first two figures of the number at the left, and in the column with the third figure at the top. Thus, the mantissa of $\log_{10} 410$ is .6128, that of $\log_{10} 417$ is .6201, and that of $\log_{10} 753$ is .8768.

Determining the characteristic of the logarithms of the numbers given in the last paragraph by the rules of article 94 we have: $\log_{10} 41 = 1.6128$, $\log_{10} 410 = 2.6128$, $\log_{10} 417 = 2.6201$, $\log_{10} 75 = 1.8751$, $\log_{10} 753 = 2.8768$. Shifting the decimal point in any number does not change the mantissa of its logarithm, and the characteristic is determined for the new number as before. Therefore $\log_{10} 4.1 = 0.6128$, $\log_{10} 4.17 = 0.6201$, $\log_{10} 7.5 = 0.8751$, $\log_{10} 7.53 = 0.8768$, $\log_{10} 7530 = 3.8768$, etc.

Shifting the decimal point again so as to convert these numbers into decimal fractions, and again determining the characteristic by the rule of article 94, we have, for $\log_{10} .41$, a characteristic -1 and

the same mantissa, .6128. The logarithm of .41 is therefore $-1 + .6128 = -.3872$. Therefore $\log_{10} .41 = -.3872$. For purposes which we shall see later, however, such a logarithm is not always written as a single negative number in this way, but the characteristic and mantissa are written as originally given. If, however, it is written -1.6128, this would indicate that the entire logarithm is negative, which is not the case. In order to indicate that the characteristic *alone* is negative, the mantissa being always positive, the complete logarithm is written $9.6128 - 10$. In this notation, it is understood that the characteristic is $9 - 10$, or -1, and that the mantissa is .6128 and is positive. Similarly, $\log .0417 = 8.6201 - 10$, $\log .00753 = 7.8768 - 10$, etc.

Summarizing these results and descriptions of the table and including the rules of article 94 for characteristics, the logarithm of any three-figure number is found from the table as follows:

(1) *Disregarding the decimal point find the mantissa corresponding to the figures of the number, on the line with the first two figures, given in the column at the left, and in the column under the third figure at the top.*

(2) *If there are figures to the left of the decimal point in the number, the characteristic is* 1 *less than the number of those figures and is positive.*

(3) *If there are no figures at the left of the decimal point in the number the characteristic is* 1 *greater than the number of zeros immediately following the decimal point and is negative.*

When it is understood that the logarithms of only one system are to be used and the base is known, it is not necessary to indicate the base in writing logarithms. Unless otherwise specified, all the logarithms used in this book are common logarithms and the base 10 will be omitted hereafter in writing.

Using the above rule, the reader should refer to the table and verify the following logarithms:

$\log 234 = 2.3692$	$\log 500 = 2.6990$
$\log .18 = 9.2553 - 10$	$\log 8.11 = 0.9090$
$\log 999 = 2.9996$	$\log 6 = 0.7782$
$\log .0126 = 8.1004 - 10$	$\log .00555 = 7.7443 - 10$
$\log 70.8 = 1.8500$	$\log .301 = 9.4786 - 10$

The mantissa of the logarithm of a four-figure number is found as follows:

Find the mantissa corresponding to the first three figures as in the rule given above. To this add the number found on the same line at the right of the page in the column with the fourth figure at the top. The characteristic is determined as before.

For example, the mantissa of log 6354 is $8028 + 3 = 8031$, and the characteristic is 3; log 6354 = 3.8031. Similarly log 635.4 = 2.8031, log .08738 = 8.9414 − 10, etc.

The nine columns at the right in the table are called *difference columns* and the process in which they are used is called *interpolation* (from *inter-polate*, "to add or introduce between").

If a logarithm is known and it is desired to find the number corresponding to it (anti-logarithm) the procedure just described is reversed. Thus to find the anti-logarithm of 1.9513, the mantissa 9513 is located in the table on the line with 89 and in the column under 4. The figures of the anti-logarithm are therefore 894. Since the characteristic is 1 there are *two* figures at the left of the decimal point in the number. Therefore, anti-log 1.9153 = 89.4. This may also be written 1.9513 = log 89.4. Similarly, anti-log 7.6599 − 10 = .00457 or 7.6599 − 10 = log .00457, 3.8176 = log 6570, etc.

In order to find the anti-logarithm of a mantissa which is not in the table, proceed as follows: Locate the next smaller mantissa in the table and write the corresponding three figures as the first three figures of the desired anti-logarithm. Subtract the mantissa located in the table from the original given mantissa and locate the difference on the same line in a difference column at the right. The figure at the top of this column is the fourth figure of the desired anti-logarithm. The decimal point is located as before, by means of the characteristic of the given logarithm.

As an example, let us find the anti-logarithm of 1.8892. The mantissa 8892 is not in the table, but the next smaller is 8887, which corresponds to the figures 774. The difference is $8892 - 8887 = 5$ and on the same line at the right this difference is in the column headed 9. The figures of the desired anti-logarithm are therefore 7749, and since the characteristic of the given logarithm is 1 there are two figures at the left of the decimal point in the anti-logarithm. The anti-logarithm of 1.8892 is therefore 77.49, or 1.8892 = log 77.49.

Similarly anti-log 4.4735 = 29,750, or 4.4735 = log 29,750, and 6.6421 − 10 = log .0004386.

97. Exercises.

Find the logarithm of each of the following numbers from the table and write it with the proper characteristic in the place indicated in the chart or on a separate sheet:

Number..	255	8.43	.746	93.64	.08765	.001111	.45670
Logarithm							

Find the number corresponding to each of the following logarithms and write it in the chart, or on a separate sheet, with decimal point properly placed:

Logarithm	1.7868	0.4265	9.9217−10	3.4584	1.2450	8.7672−10	2.8844
Number..							

Practice the use of the table with other numbers until any logarithm or anti-logarithm can be determined completely without delay or confusion.

Chapter 18

CALCULATION WITH LOGARITHMS

98. Multiplication. The object and purpose of the use of logarithms is to shorten long numerical calculations and lessen the labor required in such work. This is accomplished in multiplication by means of the principle expressed in formula (4) of article 86. This formula is (omitting the base)

$$\log (AB) = \log A + \log B,$$

where A and B are any two numbers and AB is their product.

This formula states that the logarithm of the product of any two numbers is found simply by adding their separate logarithms. With the logarithms of the product thus known, the product itself is read from the table as the anti-logarithm. Thus to find the product of 347.5 by 86.43 we find from the table log 347.5 = 2.5410 and log 86.43 = 1.9367. The sum of these two logarithms is 4.4777, and the anti-logarithm of this result is 30040. Therefore, 347.5 × 86.43 = 30040. The last figure in the product may not be exact, as we are using a four-place table, but by using a table of sufficient figures the result may be found as accurately as desired.

The operation just carried out is more directly and neatly written out as follows, in which form it is also more easily followed:

$$\begin{array}{lrl} & \log 347.5 &= 2.5410 \\ & \underline{(+)\log 86.43} &\underline{= 1.9367} \\ \text{Adding,} & \log \text{Prod.} &= 4.4777 \\ & \text{Prod.} &= 30040. \end{array}$$

The plus sign is placed before the second logarithm to indicate that it is to be added to the first.

Using this method, let us find the product of 17 × 23.

$$\begin{array}{rl} \log 17 &= 1.2304 \\ \underline{(+)\log 23} &\underline{= 1\ 3617} \\ \log \text{Prod.} &= 2.5921 \\ \text{Prod.} &= 391. \end{array}$$

In the multiplication of two small numbers such as these (17 × 23) there is, of course, small saving of time or effort. But in the multiplication of numbers with several figures or the multiplication of several factors the convenience of logarithms is appreciated. Thus let us find the product of 24 × 37 × 13. According to the extension of formula (4) in article 86, this is done as follows:

$$\begin{array}{rl} \log 24 &= 1.3802 \\ (+)\log 37 &= 1.5682 \\ \underline{(+)\log 13} &\underline{= 1.1139} \\ \log \text{Prod.} &= 4.0623 \\ \text{Prod.} &= 11544. \end{array}$$

The last figure is found by mentally *interpolating* between the figures found in the difference columns.

As an example of the use of negative characteristics let us find the product of 427.5 by .1124:

$$\begin{array}{rl} \log 427.5 &= 2.6309 \\ \underline{(+)\log .1124} &\underline{= 9.0508 - 10} \\ \log \text{Prod.} &= 11.6817 - 10 \\ &= 1.6817 \\ \text{Prod.} &= 48.05. \end{array}$$

As another example, let us find the product of 23.6 × .0428 × 938.4 × .007854

$$\begin{array}{rl} \log 23.6 &= 1.3729 \\ (+)\log .0428 &= 8.6314 - 10 \\ (+)\log 938.4 &= 2.9724 \\ \underline{(+)\log .007854} &\underline{= 7.8951 - 10} \\ \log \text{Prod.} &= 20.8718 - 20 \\ &= 0.8718 \\ \text{Prod.} &= 7.444. \end{array}$$

In all work in multiplication, as well as in division, powers and roots, it is to be remembered that negative numbers do not have logarithms, and, in particular, the logarithm of a negative number MUST NOT be written as the negative of the logarithm of the number. When negative numbers are involved in calculations, the calculation is carried out by logarithms *without regard to sign*, treating all numbers

as positive, and the proper sign of the result is then found by applying the usual algebraic rules of signs. Thus to multiply $42.5 \times (-18.7)$ the product 42.5×18.7 is found as for positive numbers and after the final result is found as the anti-logarithm of the sum of the logarithms of the factors, a minus sign is placed before it.

The examples and illustrations given in this article are sufficient to explain the method and illustrate the convenience of multiplication by means of logarithms.

99. Division. The method of division by the aid of logarithms is based on formula (5) of article 87. It is (omitting the base)

$$\log\left(\frac{A}{B}\right) = \log A - \log B,$$

A and B being any two numbers and $\frac{A}{B}$ their quotient. In order, therefore, to find the quotient of any two numbers, subtract the logarithm of the divisor from that of the dividend, and find the anti-logarithm of the result.

Thus, in order to divide 810 by 324, we proceed as follows:

$$\begin{array}{rl} \log 810 = & 2.9085 \\ \underline{(-)\log 324 =} & \underline{2.5105} \\ \log \text{Quot.} = & 0.3980 \end{array}$$

$$\therefore \quad \text{Quot.} = 2.50$$

The minus sign before the logarithm of the divisor indicates that it is to be subtracted. The quotient is the anti-logarithm of 0.3980 and is read from the table in the usual manner. The quotient is, therefore, $810 \div 324 = 2.5$, or, as it is usually written, $\frac{810}{324} = 2.5$.

If the dividend or divisor (numerator or denominator) is a decimal fraction, the negative characteristic may be handled as in multiplication. As an example, let us find the quotient of $.4735 \div .0234$:

$$\begin{array}{rl} \log .4735 = & 9.6753 - 10 \\ \underline{(-)\log .0234 =} & \underline{8.3692 - 10} \\ \log \text{Quot.} = & 1.3061 \end{array}$$

$$\text{Quot.} = 20.23.$$

The subtraction is carried out algebraically as usual and in the negative column the result is $-10 - (-10) = 0$. This method, therefore,

takes care of all signs and the logarithm of the quotient automatically comes out right.

Suppose a fraction is to be divided by a number greater than 1; the procedure in this case will be made clear by the following example: Divide $\frac{.06289}{27.43}$.

$$\begin{array}{rl} \log .06289 &= 8.7986 - 10 \\ (-)\log 27.43 &= 1.4382 \\ \hline \log \text{Quot.} &= 7.3604 - 10 \\ \text{Quot.} &= .002293. \end{array}$$

One more case remains to be considered. This is the division of a number greater than 1 by a fraction. It will be made clear by the division $\frac{92.78}{.0634}$. The logarithms are:

$$\begin{aligned} \log 92.78 &= 1.9675 \\ \log .0634 &= 8.8021 - 10. \end{aligned}$$

If the subtraction is carried out as it stands, the characteristic is $-7 + 10 = 3$, which is correct. But since the object of using the -10 is to keep the characteristic positive, the characteristic of the logarithm of the numerator must be greater than 8. The characteristic 1 is therefore written as $11 - 10$, and the procedure is as follows:

$$\begin{array}{rl} \log 92.78 &= 11.9675 - 10 \\ (-)\log .0634 &= 8.8021 - 10 \\ \hline \log \text{Quot.} &= 3.1654 \\ \text{Quot.} &= 1463. \end{array}$$

In this form the negative parts cancel in subtraction and the characteristic automatically comes out right.

The method used in the last example may always be used in division when the characteristic of the logarithm of the divisor is greater than that of the logarithm of the dividend.

By combining the operations of multiplication and division, as based on the combination formula of article 87, the values of complicated expressions may be found very simply. Thus, suppose it is desired to carry out the following indicated operations:

$$\{[(236. \times .482) \div .84] \times .634\} \div (826. \times .00729).$$

To work this out step by step by ordinary multiplication and division

as in arithmetic would be a very slow and tedious performance and one very liable to error. The labor and the probability of error are much reduced by the use of logarithms. By writing the expression in the form of a fraction with all multipliers in the numerator and all divisors in the denominator, the logarithmic method of working it becomes obvious. Thus, the expression is

$$\frac{236. \times .482 \times .634}{.84 \times 826. \times .00729}.$$

The logarithm of the denominator is now the sum of the logarithms of its factors, and is to be subtracted from the logarithm of the numerator, which is the sum of the logarithms of its factors. The procedure is, therefore, as follows:

$$\begin{array}{rcl}
\log .84 & = & 9.9243 - 10 \\
(+)\log 826. & = & 2.9170 \\
(+)\log .00729 & = & 7.8627 - 10 \\
\hline
\log \text{Denom.} & = & 20.7040 - 20.
\end{array}$$

$$\begin{array}{rcl}
\log 236. & = & 2.3729 \\
(+)\log .482 & = & 9.6830 - 10 \\
(+)\log .634 & = & 9.8021 - 10 \\
\hline
\log \text{Num.} & = & 21.8580 - 20 \\
(-)\log \text{Denom.} & = & 20.7040 - 20 \\
\hline
\log \text{Quot.} & = & 1.1540 \\
\text{Quot.} & = & 14.26.
\end{array}$$

In this case the *values* of the numerator and denominator do not need to be found, only their *logarithms* being needed. The characteristics are all made positive and the adjustment, or combination of positives and negatives, is made in the last step.

This example will be sufficient to make clear the method of handling any example in combined multiplication and division.

100. Powers. The examples of the last two articles indicate the great convenience attending the use of logarithms in arithmetical operations. It is possible, however, to perform any multiplication or division by arithmetical methods. In raising numbers to powers and in root extraction, however, the operations required are not always possible by arithmetical methods and are nearly always much more

troublesome, and it is here that the full power and simplicity of the logarithmic method are most appreciated.

Thus the operation indicated by $(11)^2$ is simple. This is not the case however with $(123.4)^8$, and the operation indicated by $(46.73)^{2.3}$ is not possible by the ordinary method of arithmetic. Similarly the operation indicated by $\sqrt{225}$ is simple but $\sqrt[3]{1237.8}$ is not, and $\sqrt[3.5]{82.4}$ is impossible by any single direct operation.

The operation of raising any real number to *any* power by the use of logarithms is based on the formula (6) of article 88, which, without indicating the base, is

$$\log (A^p) = p \log A,$$

A being any number and p the exponent of any power to which it is desired to raise A. This states that the logarithm of any power of a number is found simply by multiplying the logarithm of the number by the exponent of the power. With the logarithm of the power thus known, the value of the power itself is read from the table as an anti-logarithm.

The following example illustrates the application of the formula to a simple case. Find the cube of 21, that is, 21^3.

$$\begin{aligned} \log 21 &= 1.3222 \\ \log \text{Power} = 3 \times \log 21 &= 3.9666 \\ \text{anti-log} = \text{Power} &= 9261. \end{aligned}$$

That is, $21^3 = 9261$.

Similarly, to find $73.1^{2.5}$, we have:

$$\begin{aligned} \log 73.1 &= 1.8639 \\ 2.5 \times \log 73.1 &= 4.6597 \\ \text{Power} &= 45680. \end{aligned}$$

That is, $73.1^{2.5} = 45680$.

In such an operation as $.1382^2$, where the logarithm $9.1405 - 10$ has a negative characteristic the procedure is as follows:

$$\begin{aligned} \log .1382 &= 9.1405 - 10 \\ 2 \times \log .1382 &= 18.2810 - 20 \\ &= 8.2810 - 10 \\ \text{Power} &= .0191. \end{aligned}$$

Similarly, to find, $.00193^{1.7}$, we have:

$$\begin{aligned} \log .00193 &= 7.2856 - 10 \\ 1.7 \times \log .00193 &= 12.3855 - 17 \\ &= 5.3855 - 10 \\ \text{Power} &= .0000243. \end{aligned}$$

If the multiplication of the logarithm by the exponent is a long and tedious operation, this may itself be carried out logarithmically. The method will be sufficiently clear from the following example: Find the 4.72 power of 14.56. This is $14.56^{4.72}$ and the procedure is:

$$\begin{aligned} \log 14.56 &= 1.1632 \\ \log(\log 14.56) &= \log 1.1632 \\ &= \left.\begin{matrix} 0.0657 \\ 0.6739 \end{matrix}\right\} \\ (+)\log 4.72 & \\ \hline \log \text{Prod.} = \log(\log \text{Power}) &= 0.7396 \\ \text{Prod.} = \log \text{Power} &= 5.4900 \\ \text{Power} &= 309{,}000. \end{aligned}$$

Therefore, $14.56^{4.72} = 309{,}000$.

This example illustrates another point in the use of logarithms. In the table, the anti-logarithm of 5.4900 is read as 3090 but with characteristic 5 there must be six figures to the left of the decimal point. Two zeros are therefore added to the 3090, giving 309,000.

If a number is to be raised to a negative power its logarithm is multiplied by the negative exponent and the *entire* resulting logarithm is negative. This must be expressed with a positive mantissa in order to find the anti-logarithm (the power) in the table. Thus let us find the value of $(37.3)^{-2}$.

$$\begin{aligned} \log 37.3 &= 1.5717 \\ -2 \times \log 37.3 &= -3.1434. \end{aligned}$$

Here the minus sign before the logarithm 3.1434 indicates that the entire logarithm, and not only the characteristic, is negative. In this case, 10 is added to the logarithm *as a whole* and not to the characteristic alone, and the subtraction of 10 is indicated as usual by writing -10 after the result. This does not change the result, because algebraically

$$-3.1434 = 10 - 3.1434 - 10 = 6.8566 - 10.$$

The mantissa is now positive, while the characteristic is 6 − 10. The entire operation is, therefore, as follows:

$$\begin{array}{rl} \log 37.3 = & 1.5717 \\ -2 \times \log 37.3 = & \quad -3.1434 \\ & \quad +10.0000 - 10 \\ \hline \log \text{Power} = & 6.8566 - 10 \\ \text{Power} = & .0007188. \end{array}$$

Therefore $37.3^{-2} = .0007188$.

This result is readily understood when it is remembered that by the law of the negative exponent $37.3^{-2} = \dfrac{1}{37.3^2}$ and $37.3^2 = 1391.3$. Therefore, $37.3^{-2} = \dfrac{1}{1391.3} = .0007188$.

When a number is small and its negative exponent is small and integral, the involution may be carried out by arithmetic as just shown, but when either is large or the negative exponent is fractional it is necessary to use logarithms. In this case, the complete operation and its meaning will be understood from the preceding example.

If the exponent is negative and the characteristic of the logarithm of the number is also negative while the mantissa is positive, the negative characteristic and positive mantissa may be multiplied separately by the negative exponent and the products then combined, and the logarithm then expressed with positive mantissa as above, or the entire logarithm may be made positive by adding and subtracting 10, as usual, and then multiplying by the negative exponent to convert later to positive mantissa.

101. Roots. As any root may be expressed as a fractional power, according to the principles of Chapter 7, in order to find any root of a number by means of logarithms, it is only necessary so to express the root and use the method of the preceding article.

An example will make the method clear. Let us find the cube root of 6859. This is $\sqrt[3]{6859} = (6859)^{1/3}$ and, therefore, the logarithm of the root is one-third the logarithm of the number 6859. We have, therefore,

$$\log 6859 = 3.8363$$

$$\log \text{Root} = \tfrac{1}{3} \log 6859 = 1.2788$$

$$\therefore \quad \text{Root} = \text{anti-log} = 19.$$

Here the logarithm of the number is multiplied by $\frac{1}{3}$ or *divided* by 3. This is in accordance with the formula (7) of article 89, which, without the base, is

$$\log \sqrt[q]{A} = \frac{\log A}{q}.$$

In general, therefore, to find *any* root of any number, divide the logarithm of the number by the index of the root and read from the table the anti-logarithm of the result.

Find the 9th root of 87560, that is $\sqrt[9]{87560}$:

$$\log 87560 = 4.9423$$

$$\log \text{Root} = (\log 87560) \div 9 = 0.5491$$

$$\text{Root} = 3.541.$$

By the usual method of arithmetic, this root could be found as the cube root of the cube root, but the operation would be long and tedious. Compared with such an operation, the logarithmic method is simplicity itself.

As in the case of powers, the division of the logarithm of the number by the index of the root may also be carried out logarithmically. Thus to find $\sqrt[3.71]{43570}$, we have:

$$\begin{aligned} \log 43570 &= 4.6392 \\ \log (\log 43570) &= \log 4.6392 \\ &= 0.6664 \\ (-)\log 3.71 &= 0.5694 \\ \hline \log (\log \text{Root}) &= 0.0970 \\ \log \text{Root} &= 1.2502 \\ \text{Root} &= 17.79. \end{aligned}$$

Therefore $\sqrt[3.71]{43570} = 17.79$.

The case of a negative root index or a negative characteristic is handled in the same manner as in the case of powers.

102. The Slide Rule. The logarithmic calculations described and illustrated in this chapter are performed mechanically by means of an instrument called the *slide rule*. This instrument consists of two flat strips of wood about 10 inches long and about $\frac{1}{2}$ and $1\frac{1}{4}$ inches wide, graduated and marked in a manner somewhat similar to that of an ordinary ruler. The two strips are grooved so that one fits into and

slides along the other, with their flat faces flush. The narrower strip is called the *slide* and the wider piece the *stock* of the rule, and by sliding the slide along in its groove, any number or line on the slide may be set at any number or line on the stock. A small sliding block called the *runner*, which bears a pointer or *indicator*, also slides along the stock and is used in *setting* and reading the numbers on the scales.

The markings on the scales represent the decimal numbers 1 to 10, or 10–100, 0.1–1.0, etc., while the lengths of the divisions represent the *logarithms* of those numbers. The sections of the scales are subdivided and marked so that any number on the rule may be read to three or four figures.

If the runner indicator is set at a certain number on the stock, whose distance from the beginning or *index* of the scale represents the logarithm of that number, and the slide is set with its index at the indicator, then any number on the slide is at a distance from the stock index which is equal to the *sum* of the logarithmic lengths of the two numbers. But, according to (4), article 86, this is the logarithm of their *product*. By setting the slide index at any number on the stock, therefore, any second number on the slide is in line with their product on the stock. Thus the product of any two numbers may be read on the slide rule at a single setting and without any writing. By reversing the procedure just described, the quotient of any two numbers can be read at a single setting.

Similarly, in accordance with (6), article 88, and (7), article 89, by doubling or trebling the scale length corresponding to any number, its square or cube can be read, and, by taking a half or third of the scale length, the square or cube root can be found. Additional scales are specially marked on the slide rule according to this principle for reading roots and powers.

The slide rule thus briefly described is known as the Mannheim Slide Rule and is standard all over the world. There are other forms of slide rules which are modifications of the Mannheim rule. They have the same scales as the Mannheim, and in addition other scales for various kinds of special calculations.*

* Full and detailed descriptions of all the usual forms of slide rules and complete instructions in their use, together with the history of logarithms and the slide rule, are given in the author's "Manual for the Slide Rule," published by D. Van Nostrand Company, Princeton, N.J.

103. Exercises.

Carry out the indicated calculations in each of the following by means of logarithms:

1. $79 \times 470 \times .982$.
2. $9503 \times .008658$.
3. 137500×8799000.
4. $8075 \div 3649$.
5. $.00542 \div .04708$.
6. $2462 \div .0541$.
7. $87.3 \times .565 \div 23.45$.
8. $\dfrac{.643 \times 7095}{9 \times 67 \times .462}$.
9. $\dfrac{9097 \times 5.4086}{225 \times 593 \times .8665}$.
10. $(2.388)^5$.
11. $(1.014)^{24}$.
12. $8^{3/4}$.
13. $(.57)^{-4}$.
14. $(5648)^2 \times (.94)^3$.
15. $\sqrt{67.54}$.
16. $\sqrt[3]{2197000}$.
17. $(94.85)^{3.64}$.
18. $\sqrt[2.49]{534600}$.
19. $\sqrt{\frac{756}{229}}$.
20. $\left(\dfrac{1172}{84.6}\right)^3$.

Chapter 19

EXPONENTIAL EQUATIONS

104. Exponential Equations. An *exponential equation* was defined in article 57 as an equation in which the unknown quantity appears as the exponent of a known number. The equation

$$4^x - 16 = 0$$

is an exponential equation, the unknown quantity x being the exponent of the number 4.

In order to solve this equation, the 16 is transposed, giving

$$4^x = 16. \tag{13}$$

To *solve* the equation for x means to find the exponent of the power to which 4 must be raised to equal 16. Of course, it is known at once that this exponent is 2, for $4^2 = 16$. The solution of the equation (13) is, therefore, found at once to be $x = 2$.

The solution of the exponential equation

$$4.5^x = 17.3 \tag{14}$$

is, however, not so readily found, for it is not known in advance what power of 4.5 equals 17.3. In order to see how to solve this equation, let us first see how (13) could be solved if x were not known by inspection.

The equation (13) states that 4^x and 16 are the same. They, therefore, have the same logarithm, that is,

$$\log (4^x) = \log 16.$$

But, according to formula (6), article 88,

$$\log (4^x) = x(\log 4).$$

Therefore,

$$x(\log 4) = (\log 16).$$

In this equation x is the unknown and $(\log 4)$ is its coefficient. Therefore, dividing the equation by the number log 4,

$$x = \frac{(\log 16)}{(\log 4)}.$$

This expression does not mean that 16 is to be divided by 4, for 16/4 is 4 while we know that x is 2; neither does it mean that log 4 is to be subtracted from log 16, for log 16 − log 4 means the same thing as dividing 16 by 4. The expression $\dfrac{(\log 16)}{(\log 4)}$ means that the number which is the logarithm of 16 is to be divided by the number which is the logarithm of 4. The solution of (13) is therefore

$$x = \frac{\log 16}{\log 4} = \frac{1.2042}{0.6021} = 2,$$

which is of course correct.

It is now seen at once that the solution of (14) is

$$x = \frac{\log 17.3}{\log 4.5} = \frac{1.2380}{0.6532} = 1.895.$$

The division of 1.2380 by .6532 may be carried out with logarithms, *as a separate operation*, having nothing to do with the original solution of the equation. Thus, for the *solution of the equation* we have,

$$x = \frac{1.238}{.6532},$$

while to compute the numerical value of x we have:

$$\begin{array}{rl} \log 1.238 = & 10.0927 - 10 \\ \underline{(-)\log .6532 =} & \underline{9.8150 - 10} \\ \log \text{Quot.} = \log x = & 0.2777 \\ x = & 1.895. \end{array}$$

Expressing the solution of equations of the form of (13) and (14) in general formulas, we have the following: If

$$A^x = B, \quad x = \frac{\log B}{\log A} \tag{15}$$

and

$$\log x = \log(\log B) - \log(\log A). \tag{16}$$

Thus, to solve the exponential equation

$$13.7^x = 428.5,$$

$$\begin{array}{rl} \log(\log 428.5) = \log 2.6320 = & 0.4203 \\ \underline{(-)\log(\log 13.7) = \log 1.1367 =} & \underline{0.0555} \\ \log x = & 0.3648 \\ x = & 2.316. \end{array}$$

If one or both of the logarithms in formula (15) has a negative characteristic, the positive mantissa is to be combined with it to make the entire logarithm (or logarithms) negative. The sign of x is then determined from the quotient by the usual rules of signs and the division is carried out logarithmically without regard to sign.

As an example, let us solve the equation

$$25.3^x = .786. \tag{17}$$

$$x = \frac{\log .786}{\log 25.3} = \frac{9.8954 - 10}{1.4031}.$$

In the numerator of the fraction on the right

$$9.8954 - 10 = -1 + .8954 = -.1046$$

and, therefore, $$x = \frac{-.1046}{1.4031} = -\left(\frac{.1046}{1.4031}\right).$$

The division of the positive numbers inside the parentheses is carried out in the usual way, and the minus sign placed before the quotient. The solution of equation (17) is, therefore,

$$x = -.07454.$$

The solution of an exponential equation with the unknown appearing as a term in a polynomial exponent, such as

$$(7.52)^{(x-2)} = 327.4,$$

will be understood from the following solution of this equation:

$$\log (7.52)^{(x-2)} = \log 327.4,$$

or, $$(x - 2) \log 7.52 = \log 327.4$$

$$x \log 7.52 - 2 \log 7.52 = \log 327.4$$

$$x \log 7.52 = 2 \log 7.52 + \log 327.4.$$

$$\therefore \quad x = \frac{2 \log 7.52 + \log 327.4}{\log 7.52}$$

$$= \frac{2(.8762) + 2.5151}{.8762}$$

$$= \frac{1.7524 + 2.5151}{.8762}$$

or, $$x = \frac{4.2675}{.8762} = 4.870,$$

the division at the last being performed logarithmically.

105. Compound Interest. Interesting and useful applications of exponential equations and logarithms are the calculation of the amount of a sum of money at compound interest, the interest rate and time to run being known, and the inverse problem of finding the principal required to produce a specified amount in a specified time, the rate being known. The logarithmic method is used in banking and insurance, the accuracy obtained depending on the size of the table used.

Let P represent the principal in dollars, and r the interest rate in per cent, that is, number of cents per dollar, and let n represent the number of years it is to run. Then if the interest is compounded annually, the interest at the end of the first year is $P \times r$, or Pr, and the amount is the sum of the principal and interest, $P + Pr$. Taking out the common factor P in this binomial, the amount is

$$P(1 + r).$$

This states that the amount at the end of one year is found by multiplying the principal by $1 + r$.

The amount at the end of the first year is the principal for the second year, and the amount at the end of the second year is this principal, $P(1 + r)$, multiplied by $(1 + r)$, as above. That is, at the end of the second year the amount is

$$[P(1 + r)](1 + r) = P(1 + r)^2.$$

Similarly, at the end of the third year the amount is this principal multiplied by $(1 + r)$, or $P(1 + r)^3$; at the end of the fourth year it is $P(1 + r)^4$, and after n years the amount is

$$A = P(1 + r)^n. \tag{18}$$

If the interest is compounded semi-annually there are two interest periods in a year and the interest per dollar in one period is $\frac{1}{2}r$ or $r/2$. The amount is, therefore,

$$A = P\left(1 + \frac{r}{2}\right)^{2n},$$

where r is still the *annual* rate and n the number of years to run. If the interest is compounded quarterly, the amount after n years is $A = P\left(1 + \frac{r}{4}\right)^{4n}$; and, in general, if the number of interest periods per year is p, the amount after n years at the annual rate r is

$$A = P\left(1 + \frac{r}{p}\right)^{pn}, \tag{19}$$

where P is the original principal.

The formulas (18) and (19) are used in calculating the amount at compound interest of sums of money which run for any length of time at any given rate of interest when the interest is compounded at specified intervals.

Suppose it is desired to solve the inverse problem, that is, to find the principal necessary to produce a certain amount when the time and the rate are known. In this case P is an unknown quantity in the equations (18) and (19) with A, r, p and n known. The equations (18) and (19) are, therefore, to be solved for P.

Writing (18) as

$$P(1 + r)^n = A, \tag{20}$$

and dividing by the coefficient of P, $(1 + r)^n$, we have

$$P = \frac{A}{(1 + r)^n}, \tag{21}$$

or, using the negative exponent,

$$P = A(1 + r)^{-n}. \tag{22}$$

This formula holds in the case of annual compounding.

With p interest periods per year, equation (19) is solved for P in the same way and gives

$$P = A\left(1 + \frac{r}{p}\right)^{-pn}. \tag{23}$$

If it is desired to find the number of years a certain sum must run to produce a specified amount at a given rate, n is the unknown in equation (20) and this exponential equation must be solved for n.

Following the method of article 104, take the logarithm of both sides of equation (20). This gives, since $P(1 + r)^n$ is a product,

$$\log P + \log [(1 + r)^n] = \log A.$$

Transpose, $$\log [(1 + r)^n] = \log A - \log P.$$

Applying the power formula to $\log (1 + r)^n$, this becomes

$$n \log (1 + r) = \log A - \log P$$

$$\therefore \quad n = \frac{\log A - \log P}{\log (1 + r)}. \tag{24}$$

In this expression $\log (1 + r)$ is NOT equal to $\log 1 + \log r$, but is the logarithm of the single number which is the sum $1 + r$. In any calculation by formula (24), therefore, this sum must first be found and the logarithm taken afterward.

The equation (19) can be solved for n in the same way and the result will be found to be

$$n = \frac{\log A - \log P}{p \log \left(1 + \frac{r}{p}\right)}. \tag{25}$$

The same remarks apply here to the sum $1 + \frac{r}{p}$ as to the sum $1 + r$ in (24).

The compound interest formulas can also be solved for r as unknown but the solution is somewhat complicated, and as the interest rate is usually fixed by law (not mathematical) it is of little value.

We will give an illustration of the use of each of the formulas (18), (19), (22), (23), (24) and (25).

Example 1. Find the amount of \$2500 in 18 years at 4 per cent compounded annually.

Solution. Here $P = 2500$, $r = .04$, $n = 18$ and, therefore, by (18) the amount is

$$\begin{aligned} A &= 2500 \times (1.04)^{18} \\ \log A &= \log 2500 + 18 \log 1.04 \\ &= 3.3979 + 18 \times .0170 \\ &= 3.3979 + .3060. \\ \therefore \quad \log A &= 3.7039 \\ A &= \$5057. \end{aligned}$$

Example 2. If the interest in Example 1 is compounded quarterly, what will the amount be?

Solution. Here P, r, and n are the same as before and in formula (19) $p = 4$. Therefore,

$$\begin{aligned} A &= 2500 \left(1 + \frac{.04}{4}\right)^{(4 \times 18)} \\ &= 2500 \times 1.01^{72} \end{aligned}$$

$$\log A = \log 2500 + 72 \log 1.01$$
$$= 3.3979 + 72 \times .0043$$
$$= 3.3979 + .3096.$$
$$\therefore \quad \log A = 3.7075$$
$$A = \$5099.$$

Example 3. What principal would be required to produce \$1000 in 10 years at 5 per cent compound interest?

Solution. Here $A = 1000$, $n = 10$, $r = .05$ and the principal P is given by formula (22).

$$P = 1000(1.05)^{-10}$$
$$\log P = \log 1000 - 10 \log 1.05$$
$$= 3.0000 - 10 \times .0212$$
$$= 3.0000 - .2120.$$
$$\therefore \quad \log P = 2.7880$$
$$P = \$613.70$$

Example 4. What principal would be required in Example 3 if the interest were compounded semi-annually?

Solution. Here A, n and r are the same as before and in formula (23) $p = 2$. Therefore,

$$P = 1000\left(1 + \frac{.05}{2}\right)^{-2\times 10}$$
$$= 1000 \times 1.025^{-20}$$
$$\log P = \log 1000 - 20 \log 1.025$$
$$= 3.0000 - 20 \times .0107$$
$$= 3.0000 - .2140.$$
$$\therefore \quad \log P = 2.7860$$
$$P = \$611.$$

Example 5. How long will it take \$100 to amount to \$150 at 6 per cent compounded annually?

Solution. Here $P = 100$, $A = 150$, $r = .06$ and therefore, according to (24), the required time in years is

$$n = \frac{\log 150 - \log 100}{\log 1.06} = \frac{2.1761 - 2.0000}{.0253}$$

$$= \frac{.1761}{.0253} = 6.96 \text{ years,}$$

the division .1761/.0253 being performed logarithmically.

Example 6. If, in Example 5, the interest is compounded quarterly, what would be the time required?

Solution. Here all the quantities are the same as before and in formula (25) $p = 4$. Therefore,

$$n = \frac{\log 150 - \log 100}{4 \log \left(1 + \frac{.06}{4}\right)} = \frac{\log 150 - \log 100}{4 \log 1.015}$$

$$= \frac{2.1761 - 2.0000}{4 \times .0065}.$$

$$\therefore \quad n = \frac{.1761}{.0260} = 6.77 \text{ years.}$$

A problem which sometimes arises in compound interest calculations is the following: Given the invested principal (P), the amount to be produced (A), and the time to run or number of interest periods (p and n), what annual interest rate will be required? The solution of this problem does not require the solution of an exponential equation, but it does require a root extraction which must be performed with logarithms. The solution will therefore be given here.

Refer to formula (19) above, and divide both sides by P; this gives

$$\left(1 + \frac{r}{p}\right)^{pn} = \frac{A}{P}.$$

$$\therefore \quad 1 + \frac{r}{p} = \sqrt[pn]{\frac{A}{P}}, \quad \frac{r}{p} = \sqrt[pn]{\frac{A}{P}} - 1,$$

and

$$r = p(\sqrt[pn]{A/P} - 1). \tag{26}$$

In order to calculate r by this formula the pn root of the quotient A/P must be extracted by using logarithms, 1 is then subtracted from the root, and the remainder is multiplied by p.

Example 7. What interest rate is necessary for a principal of \$100 to amount to \$150 when compounded quarterly for five years?

Solution. Here $P = \$100$, $A = \$150$, $p = 4$ periods per year, and $n = 5$ years. Therefore $A/P = \frac{150}{100} = 1.5$, $pn = 4 \times 5 = 20$, and $\sqrt[pn]{\frac{A}{P}} = \sqrt[20]{1.5}$. To extract this root, $\log 1.5 = .1761$, $\log (\sqrt[20]{1.5}) = .1761 \div 20 = .0088$, and $\sqrt[20]{1.5} = \text{anti-log } .0088 = 1.0205$. The formula (26) then gives

$$r = p(1.0205 - 1) = 4 \times .0205 = .0820 = 8.2\%.$$

The annual rate required is therefore 8.2%.

106. Exercises and Problems.

Solve the following exponential equations:

1. $3.74^x = 19.28$.
2. $.0732^x - 4 = 0$.
3. $87.34^{x+3} = 74{,}320$.
4. $\dfrac{4.53^x}{4.53^{1.5}} = 35$.

5. A soldier of the American Revolution deposited his savings of $575 (American money) in a bank in 1783 and was killed in battle before drawing on the account. If the account has not been outlawed, what was its value in 1945 at 4% interest compounded annually?

6. A man estimates that it will take $2500 a year to give his son a four-year college course, to begin when he is 18 years old. In order to produce the necessary amount, what sum must he deposit at the boy's birth if it is to remain at 5% interest compounded annually until the boy's 18th birthday?

7. A man invests $1500 for his daughter on her 10th birthday and provides that she shall receive the principal with accrued interest at 6% compounded semi-annually, when the amount is $2500. How old will she be when she receives it?

8. A boy saves $200, including interest, in a Christmas savings account in a bank, and plans to use it six years later to pay his first quarter college tuition, which will cost $325. He plans to deposit the $200 in a compound interest account, the interest being compounded monthly. What must be the annual interest rate if he is to have the amount needed at the time planned?

9. A man signs a note for $300, payable two years from the date of the note, including interest at 6% per year, compounded quarterly. The interest is discounted and he receives the "present value" of the note in cash. How much money does he actually receive?

10. How much more on the dollar will 5% compounded semi-annually produce in 5 years than 5% simple interest?

Part 5

SPECIAL SUBJECTS

Chapter 20

RATIO, PROPORTION AND VARIATION

107. Meaning of Ratio and Proportion. The expression $\frac{a}{b}$ represents the quotient obtained by dividing the number a by the number b. The value of $\frac{a}{b}$ is, therefore, a single number. This number is called the "*ratio* of a to b." Thus the ratio of 6 to 3 is 2 and the ratio of 1 to 2 is .5. A ratio is written in the form of a fraction and as it has in one sense the same meaning as a fraction, namely, quotient of numerator by denominator, ratios have all the properties of fractions and in algebraic transformations can be treated as fractions. The notation $a : b$ used in arithmetic for ratio can be thought of as being derived from the division sign ($\div$), the dash between the two dots being omitted. Then $a : b$, means the same as $a \div b$, a/b, or $\frac{a}{b}$.

In the ratio $a : b$ or $\frac{a}{b}$ the number a is called the *antecedent* and the number b is called the *consequent*.

When the ratios $a : b$ and $c : d$, or $\frac{a}{b}$ and $\frac{c}{d}$ are equal the four numbers a, b, c, and d are said to be *in proportion*. This is written

$$\frac{a}{b} = \frac{c}{d} \quad \text{or} \quad a : b :: c : d, \tag{1}$$

the first form being an ordinary equation and having all the properties of an equation. The expression (1) may be read as an equation, "a divided by b equals c divided by d," or in the usual form for proportion, "a is to b as c is to d."

In the proportion $a : b :: c : d$ the numbers a, b, c, d are called the *terms* or *members* of the proportion. The first and last terms a and d are called the *extremes* and the intermediate terms b and c are called the *means*.

108. Solution of a Proportion. Since a proportion has the properties of an equation any one of its terms may be considered as unknown, and the proportion may be solved for this member in terms of the others. (It is to be noted that the word *member* does not have the same meaning here as previously given in connection with equations.) This solution is performed in the following manner:

Suppose the term or member a is unknown in the proportion (1). First multiply both sides of the equation, as first written, by bd. This gives $\frac{abd}{b} = \frac{bcd}{d}$ and when like factors are cancelled in numerators and denominators,

$$ad = bc. \tag{2}$$

In this equation we only have to divide by the coefficient of the unknown and obtain finally,

$$a = \frac{bc}{d}. \tag{3a}$$

Similarly (2) can be solved for b, c or d and gives

$$b = \frac{ad}{c}, \quad c = \frac{ad}{b}, \quad d = \frac{bc}{a}. \tag{3b}$$

When only three members of a proportion are known the process of determining the fourth is called *solving the proportion.*

109. Transformations of a Proportion. The equation (2) expresses a very important property of a proportion. Comparing (2) with (1) it is seen that the two terms whose product appears on the left of (2) are the extremes, and those whose product appears on the right are the means of (1). Equation (2) therefore shows that

In any proportion the product of the means equals the product of the extremes.

If the equation (2) is divided by cd and common factors are cancelled in numerators and denominators there results,

$$\frac{a}{c} = \frac{b}{d}. \tag{4a}$$

When this new proportion is compared with the original proportion $\frac{a}{b} = \frac{c}{d}$ it is seen that the places of the means b and c are interchanged.

Similarly, dividing (2) by ab and reducing, there results

$$\frac{d}{b} = \frac{c}{a}, \tag{4b}$$

in which the extremes a and d have been interchanged from (1).

The transformation of the proportion (1) into either of the forms (4), by interchanging the means or the extremes, is called *alternation* of the terms. We have, therefore, the result, that

In any proportion the means or the extremes may be alternated.

Next let us divide (2) by ac. This gives $ad/ac = bc/ac$ and on cancelling, $d/c = b/a$, or

$$\frac{b}{a} = \frac{d}{c}. \tag{5}$$

Comparing this with the original proportion (1) it is seen that both ratios have been inverted. Therefore,

Both ratios of any proportion may be inverted.

Since equation (2) is derived from (1), and (4) and (5) are derived from (2), equations (1), (4) and (5) may be derived from (2). Therefore,

If the product of one pair of numbers equals the product of another pair, a proportion may be formed with either pair of the numbers as means and the other pair as extremes.

By one of the fundamental properties of equations, any one number may be added to both sides. Let 1 be added to both sides of the equation (1). This gives

$$\frac{a}{b} + 1 = \frac{c}{d} + 1.$$

Combining the two terms of the equation on the left with the common denominator b and those on the right with the common denominator d this becomes

$$\frac{a+b}{b} = \frac{c+d}{d}. \tag{6}$$

That is, "if a is to b as c is to d, then $(a + b)$ is to b as $(c + d)$ is to d." The transformation of the proportion from (1) to (6) is called *composition*, and the statement just made in quotation marks is expressed by saying that

The terms of a proportion are in proportion by composition.

If instead of being added to both sides of equation (1) the number 1 is subtracted from both sides, and the same combination is performed, there results then,

$$\frac{a-b}{b} = \frac{c-d}{d}. \tag{7}$$

That is, "if a is to b as c is to d, then $(a - b)$ is to b as $(c - d)$ is to d." This transformation is called *division* of the proportion. Therefore,

The terms of a proportion are in proportion by division.

Next, let us divide equation (6) by equation (7). This gives

$$\frac{\left(\frac{a+b}{b}\right)}{\left(\frac{a-b}{b}\right)} = \frac{\left(\frac{c+d}{d}\right)}{\left(\frac{c-d}{d}\right)},$$

or, by inverting the divisors and multiplying the fractions,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}. \tag{8}$$

This states that "if a is to b as c is to d, then $a + b$ is to $a - b$ as $c + d$ is to $c - d$." This is expressed by the statement that

The terms of a proportion are in proportion by composition and division.

The proportion (8) can be alternated like the original proportion (1), giving

$$\frac{a+b}{c+d} = \frac{a-b}{c-d}. \tag{9}$$

Similarly the various other transformations applied to (1) may also be applied to (8).

Suppose the several ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}$, etc., are all equal. They are then all equal to the same thing, say r, and we can write

$$\frac{a}{b} = r, \quad \frac{c}{d} = r, \quad \frac{e}{f} = r, \quad \frac{g}{h} = r, \text{ etc.}$$

$$\therefore \quad a = rb,\ c = rd,\ e = rf,\ g = rh, \text{ etc.}$$

Adding, $$a + c + e + g = r(b + d + f + h).$$

$$\therefore \quad \frac{a+c+e+g}{b+d+f+h} = r.$$

But $r = \frac{a}{b}$, $r = \frac{c}{d}$, etc. Therefore

$$\frac{a+c+e+g}{b+d+f+h} = \frac{a}{b}, \tag{10}$$

or,

$$\frac{a+c+e+g}{b+d+f+h} = \frac{c}{d}, \text{ etc.}$$

Now, the terms of the polynomial numerator in these expressions are the antecedents of the equal ratios $\frac{a}{b}$, $\frac{c}{d}$, etc., and the terms of the polynomial denominator are the consequents. Therefore,

In a series of equal ratios, the ratio of the sum of all the antecedents to the sum of all the consequents is equal to any one of the original ratios.

110. Continued Proportional and Mean Proportional. If the numbers a, b, c, d, etc., are so related that

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots,$$

they are said to form a *continued proportion*. Any two equal ratios such as those we have been discussing, forming a proportion like (1), are said in distinction to form a *simple proportion*.

If any three numbers a, b and c are in continued proportion, then, as above

$$\frac{a}{b} = \frac{b}{c}, \tag{11}$$

and in this proportion the means are equal. In this case, the number b is said to be a *mean proportional* between a and c. A mean proportional is also called a *geometric mean* (see article 116).

Now, the product of the means in the proportion (11) equals the product of the extremes. Therefore,

$$b^2 = ac.$$

$$\therefore \quad b = \sqrt{ac}. \tag{12}$$

This states that *a mean proportional between two numbers equals the square root of their product.* Also, *if a number equals the square root of the product of two others it is their mean proportional.*

111. Meaning of Variation. If any two variables are so related that their ratio remains constant (unchanged) when their values separately change they are said to *vary directly*. Let x and y be the variables and $\frac{y}{x}$ their ratio. Then, since $\frac{y}{x}$ is also a fraction, the numerator and denominator may both be multiplied by the same number without changing the value of the fraction. That is, if

$$\frac{y}{x} = c, \tag{13}$$

then, also,

$$\frac{ny}{nx} = c,$$

where c is the constant value of the ratio.

From (13) we have

$$y = cx. \tag{14a}$$

If the value of the constant c is not known, this relation is written

$$y \propto x, \tag{14b}$$

the symbol $\propto$ is the symbol of variation and either (14a) or (14b) may be expressed in words by saying that "y varies directly as x." The form (14b) is called a *variation* and the form (14a) is called the equation of the variation (14b). In either form (14) the variable x is called the *independent variable* and y the *dependent variable.*

When y varies *directly* as x, as in (14), any change in x is accompanied by a corresponding change in y. In particular, if x increases, y also increases, and if x decreases, so does y. If, on the other hand, y decreases when x increases and increases when x decreases, y is said to vary *inversely* as x. This is expressed by writing

$$y \propto \frac{1}{x}. \tag{15b}$$

When y varies inversely as x, it is their product which remains constant, instead of their ratio. That is,

$$xy = c, \tag{16}$$

where c is not necessarily the same as in (13). This expression clearly shows the relation between x and y, for when either increases, the other must decrease proportionally in order that their product shall remain the same, that is, they vary not in the same way but in the inverse sense.

Solving (16) for y, we have

$$y = \frac{c}{x}, \tag{15a}$$

and either this form or (15b) may be used to express inverse variation, just as either form (14) may be used to express direct variation. In (15), as in (14), x and y are the independent and dependent variables, respectively.

If the variable y varies directly as x and at the same time varies

inversely as a third variable z the relation between the three variables is expressed by writing

$$y \propto \frac{x}{z}, \quad \text{or} \quad y = c \cdot \frac{x}{z}. \tag{17}$$

If y varies directly as x and also directly as z, y is said to vary *jointly* as x and z. This is written

$$y \propto xz, \quad \text{or} \quad y = cxz. \tag{18}$$

In (17) and (18) both x and z are independent variables and y is the dependent variable.

112. Variation and Proportion. Suppose that y varies directly as x, and when x has the particular fixed value x_1, y has the particular fixed value y_1. The particular values of the variables are called *corresponding values* of x and y and according to (14*a*),

$$y_1 = cx_1.$$

Let x_2 and y_2 be another pair of corresponding values of x and y. Then, also,

$$y_2 = cx_2.$$

Dividing these two equations, member by member,

$$\frac{y_1}{y_2} = \frac{cx_1}{cx_2},$$

and, since c is the same in both equations, it may be cancelled. Therefore,

$$\frac{y_1}{y_2} = \frac{x_1}{x_2}, \tag{19}$$

and this is a simple proportion. We have, therefore, the following very important result:

When one variable varies directly as another, two pairs of their corresponding values are in proportion.

This relation provides a very useful method of determining values of measurable quantities by calculation. Suppose x and y represent any two measurable physical quantities or dimensions and it is known that they vary proportionally. Then, if a pair of corresponding values of the two is determined in some way (x_1, y_1), it is thereafter only necessary to measure one of the variables in order to determine the corresponding value of the other, for the other can be calculated by solving the proportion (19).

From (15*a*) we have for corresponding values of the variables in inverse proportion,

$$y_1 = \frac{c}{x_1}, \quad \text{and} \quad y_2 = \frac{c}{x_2}.$$

Dividing,

$$\frac{y_1}{y_2} = \frac{\left(\frac{c}{x_1}\right)}{\left(\frac{c}{x_2}\right)} = \frac{c}{x_1} \cdot \frac{x_2}{c}.$$

$$\therefore \quad \frac{y_1}{y_2} = \frac{x_2}{x_1}. \tag{20}$$

In this case y and x are said to be in *inverse proportion.* In distinction the proportion (19) is called a *direct proportion.*

Similarly, from (17),

$$y_1 = \frac{cx_1}{z_1}, \quad y_2 = \frac{cx_2}{z_2}.$$

Dividing,

$$\frac{y_1}{y_2} = \frac{\left(\frac{cx_1}{z_1}\right)}{\left(\frac{cx_2}{z_2}\right)} = \frac{cx_1}{z_1} \cdot \frac{z_2}{cx_2}.$$

$$\therefore \quad \frac{y_1}{y_2} = \frac{x_1 z_2}{x_2 z_1}. \tag{21}$$

In this case y is said to be directly proportional to x and inversely proportional to z.

From (18),

$$y_1 = cx_1 z_1, \quad y_2 = cx_2 z_2.$$

Dividing,

$$\frac{y_1}{y_2} = \frac{x_1 z_1}{x_2 z_2}, \tag{22}$$

and y is said to be directly proportional to both x and z.

The algebraic transformations of article 115 apply to all the proportions (19) to (22).

The proportions (20), (21) and (22) are useful for the same reason that (19) is useful and apply in cases where two or three variables are related to one another according to (15), (17) and (18). For this reason the subjects of variation and proportion are of great importance in many branches of science and engineering.

Chapter 21

PROGRESSIONS AND SERIES

113. Arithmetical Progression. If a sequence of numbers is such that the difference between every two consecutive numbers in the sequence is the same, the numbers are said to be in *arithmetical progression.* Thus, 7, 10, 13, 16, 19, . . . form an arithmetical progression in which the difference between consecutive numbers is 3. The numbers in an arithmetical progression are called the *terms,* and the difference between consecutive terms is called the *common difference* of the progression.

If a denotes the first term of an arithmetical progression and d the common difference, then the second term is $a + d$, the third term is $(a + d) + d$, or $a + 2d$. Similarly the fourth term is $a + 3d$, etc. The sequence of terms is therefore

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad a + 4d, \text{ etc.}$$

In this sequence it is to be noted that any particular term is found by adding to the first term a a number which is the common difference d multiplied by the *number of the term less* 1. Thus the fifth term is $a + 4d$, the eleventh is $a + 10d$, and in general the nth term is $a + (n - 1)d$. The terms in any arithmetical progression are therefore

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \ldots, \quad a + (n - 1)d, \ldots . \qquad (23)$$

114. Properties of an Arithmetical Progression. If n is the total number of terms in an arithmetical progression the nth term is the last and if we let l denote the last term its value according to (23) above, is

$$l = a + (n - 1)d. \qquad (24)$$

Thus, suppose a certain arithmetical progression begins with the number 4 and has 17 terms. If the common difference is $2\frac{1}{2}$, what is the last term?

We have $a = 4$, $n = 17$, $d = 2\frac{1}{2}$. The last term is, therefore,

$$l = 4 + (17 - 1) \times 2\tfrac{1}{2} = 44.$$

Next let us try to find the sum S of any number of terms of an arithmetical progression. Evidently, according to (23), the sum may be expressed as

$$S = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + [a + (n - 1)d].$$

It is also evident that if we start with the last term and *subtract* the common difference, the next-to-last term is $l - d$, the second-from-last is $l - 2d$, the next is $l - 3d$, and so on, until finally we get back to the first term, which is found by transposing (24) to be

$$a = l - (n - 1)d.$$

The terms of the sequence can, therefore, be written in descending order as

$$l, l - d, l - 2d, l - 3d, \ldots, l - (n - 1)d,$$

and, therefore, the sum of the terms may also be written

$$S = l + (l - d) + (l - 2d) + (l - 3d) + \ldots + [l - (n - 1)d].$$

Writing the two expressions for S together,

$$S = a + a + d + a + 2d + a + 3d + \ldots + a + (n - 1)d$$
$$S = l + l - d + l - 2d + l - 3d + \ldots + l - (n - 1)d.$$

Adding these two equations member by member, there results,

$$2S = (a + l) + (a + l) + (a + l) + \ldots + (a + l),$$

and there are n of these expressions in parentheses. That is,

$$2S = n(a + l).$$

$$\therefore \quad S = \frac{n}{2}(a + l). \tag{25}$$

This states that the sum of the terms of an arithmetical progression is found by multiplying half the number of terms by the sum of the first and last terms.

Formula (25) expresses the sum S in terms of a, n, and l. If it is desired to express S in terms of a, n, and d the value of l from (24) may be substituted in (25). This gives

$$S = \frac{n}{2}\{a + [a + (n - 1)d]\}.$$

Simplifying,

$$S = n\left[a + \frac{d}{2}(n - 1)\right]. \tag{26}$$

By means of formulas (24), (25) and (26) any one of the five numbers a, d, n, l, S may be expressed in terms of any other three. If, therefore, in any arithmetical progression any three of these five numbers are given, the other two may be found by using these formulas.

As an example suppose that in a certain raffle or lottery the tickets are numbered consecutively from 1 to 50 and the price of each ticket in cents is equal to its number. What are the proceeds of the lottery?

Here the first term is $a = 1$, the number of terms is $n = 50$, and the last term is $l = 50$. The sum, according to formula (25), is, therefore,

$$S = \frac{50}{2}(1 + 50) = 25 \times 51 = 1275¢$$

or,

$$S = \$12.75.$$

Again, if the last term in a certain progression is $l = 1\frac{1}{2}$, the common difference $d = \frac{1}{2}$ and the sum $S = -7\frac{1}{2}$, find the first term and the number of terms, and write out the progression.

Here we have two unknowns, a and n, and must, therefore, have two equations in order to find their values. These equations are furnished by the formulas (24), (25) and (26). In (24) and (25) a and n are the only unknowns. Substituting in these two the given values of l, d and S, we have

$$1\tfrac{1}{2} = a + \tfrac{1}{2}(n - 1)$$

$$-7\tfrac{1}{2} = \frac{n}{2}(a + 1\tfrac{1}{2}),$$

which are simultaneous equations to be solved for a and n.

Clearing fractions and rearranging the terms these equations are

$$2a + n = 4$$

$$2na + 3n = -30.$$

From the first,

$$a = \frac{4 - n}{2}. \tag{27}$$

Substituting this in the second equation and transposing,

$$2n\left(\frac{4 - n}{2}\right) + 3n + 30 = 0.$$

Simplifying,

$$n^2 - 7n - 30 = 0.$$

This is a quadratic equation with n as unknown. Factoring,

$$(n - 10)(n + 3) = 0.$$

$$\therefore \quad n = 10, \quad n = -3.$$

Obviously there cannot be -3 terms. Therefore, the number of terms is 10.

Putting $n = 10$ in (27), we have

$$a = \frac{4 - 10}{2} = -3,$$

which is the first term.

The first term of the progression is, therefore, -3, the common difference is $\frac{1}{2}$, and there are 10 terms. The complete sequence of numbers is, therefore,

$$-3, \quad -2\tfrac{1}{2}, \quad -2, \quad -1\tfrac{1}{2}, \quad -1, \quad -\tfrac{1}{2}, \quad 0, \quad \tfrac{1}{2}, \quad 1, \quad 1\tfrac{1}{2}.$$

It is seen that the last term is $1\frac{1}{2}$, as given, and if the terms are added the sum will be found to be $-7\frac{1}{2}$, also as given.

115. Arithmetical Means. If three numbers form an arithmetical progression the middle term is called the *arithmetical mean* of the other two. If the three numbers are a, b, c, then by the definition of an arithmetical progression the difference between a and b is the same as the difference between b and c. That is,

$$b - a = c - b.$$

Transpose,

$$2b = a + c.$$

$$\therefore \quad b = \frac{a + c}{2}. \tag{28}$$

The arithmetical mean of two numbers is half their sum.

The number b in (28) is sometimes called the *average* of the two numbers a and c. Similarly the average or *mean value* of three numbers, a, b, c, is $\dfrac{a + b + c}{3}$, and, in general, the mean value of n numbers, $a_1, a_2, a_3, a_4 \ldots, a_n$, is

$$m = \frac{1}{n}(a_1 + a_2 + a_3 + \ldots + a_n), \tag{28a}$$

that is, their sum divided by their number.

The true average or mean of a set of numbers is defined and studied

in connection with the *theory of probability and mean values*, in higher mathematics and statistics, and is of great importance in insurance, physics and chemistry.

116. Geometrical Means. Besides the arithmetical mean defined in (28) above, there is another form of mean value which is defined by the formula

$$b = \sqrt{ac}. \tag{29}$$

This value of b is called the *geometrical mean* of the numbers a and c, as stated in article 110, where this formula was derived (formula (12)).

The name "geometrical mean" comes from a property of the circle found in geometry, namely, that if a line is drawn from a point on the circumference of a circle perpendicular (at right angles) to any diameter, its length is a mean proportional between the lengths of the two segments into which it divides the diameter.

The formula (29) above (formula (12), article 110) was derived from formula (11), which is

$$\frac{a}{b} = \frac{b}{c}, \tag{30}$$

By means of this formula any number of geometrical means can be inserted between the numbers a and c. For, if x is the geometrical mean of a and y, then by (30),

$$\frac{a}{x} = \frac{x}{y};$$

if y is the geometrical mean of x and z,

$$\frac{x}{y} = \frac{y}{z};$$

and if z is the geometrical mean of y and c,

$$\frac{y}{z} = \frac{z}{c}.$$

When these three equations are combined there results the *continued proportion*

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{z} = \frac{z}{c}, \tag{31}$$

in which the three numbers x, y, z are all geometrical means of the two numbers a and c.

Now the continued proportion (31) states that the ratio of any two

consecutive numbers in the sequence a, x, y, z, c is the same as that of any other two consecutive numbers in the sequence. When the numbers of a sequence are related in this manner they are said to be in *geometrical progression.*

117. Geometrical Progression. From the above we have the following

DEFINITION: *A geometrical progression is a sequence of numbers in which the ratio of every two consecutive numbers is the same.*

Thus 3, 6, 12, 24, 48, etc., form a geometrical progression because

$$\tfrac{3}{6} = \tfrac{6}{12} = \tfrac{12}{24} = \tfrac{24}{48} = \cdots,$$

or,

$$\tfrac{48}{24} = \tfrac{24}{12} = \tfrac{12}{6} = \tfrac{6}{3} = 2.$$

In this sequence the ratio of consecutive numbers is 2. This is the multiplier by which each number in the sequence is multiplied to give the next number. This ratio is called the *common ratio* of the geometrical progression. The numbers in the sequence are called the *terms* of the progression.

A geometrical progression, in which each term is formed from the preceding term by *multiplication* by the common ratio, is to be distinguished from an arithmetical progression, in which each term is formed from the preceding term by *addition* of the common difference.

In the geometrical progression given above, with first term 3 and common ratio 2, the second term 6 is equal to 3×2. The third term is $12 = 6 \times 2 = (3 \times 2) \times 2 = 3 \times 2^2$; $24 = 12 \times 2 = (3 \times 2^2) \times 2 = 3 \times 2^3$; $48 = 3 \times 2^4$; $96 = 3 \times 2^5$; etc.

Similarly, if the first term is a and the common ratio is r, the terms of a geometrical progression will be

$$a,\ ar,\ (ar)\cdot r = ar^2,\ (ar^2)\cdot r = ar^3,\ ar^4,\ ar^5, \text{ etc.}$$

It is to be noted here that any particular term is formed by multiplying the preceding term by r, or by multiplying the first term a by r raised to a power which is equal to the *number of the term less* 1. Thus the second term is ar^1, the third is ar^2, the fourth is ar^3, the 10th would be ar^9, and in general the nth term would be ar^{n-1}.

The terms of a complete geometrical progression with first term a and common ratio r are, therefore,

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^{n-1},\ \ldots. \tag{32}$$

The progression is said to be *increasing* or *decreasing* according as r is

greater or less than 1, for if r is greater than 1, each term is greater than the preceding term, while if r is a fraction, each term is smaller than the preceding term.

118. Properties of a Geometrical Progression. If the total number of terms in a geometrical progression is n, then the nth term is the last term l and, therefore, from (32),

$$l = a \cdot r^{n-1}. \tag{33}$$

Suppose each one of a certain sequence of numbers is formed by multiplying the preceding number by 2, and the first number is $3\frac{1}{2}$. If there are 13 numbers in the sequence, what is the last?

Here the first term is $a = 3\frac{1}{2}$, the common ratio is $r = 2$, and the number of terms is $n = 13$. The last term, according to (33), is, therefore,

$$l = 3\tfrac{1}{2} \times 2^{12} = 3\tfrac{1}{2} \times 4096 = 14{,}336.$$

The sum S of all the terms of a geometrical progression of n terms, with first term a and common ratio r is, according to (32),

$$S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}. \tag{34}$$

If both sides of this equation are multiplied by r, term by term, the next-to-last term being ar^{n-2}, it becomes

$$rS = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n.$$

Write this equation together with (34),

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\ rS &= \phantom{a + {}} ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n. \end{aligned}$$

Subtract the second from the first, member by member, and there results,

$$S - rS = a - ar^n.$$

Factor,

$$S(1 - r) = a(1 - r^n).$$

$$\therefore \quad S = \frac{a(1 - r^n)}{1 - r}. \tag{35}$$

If r is greater than 1, $1 - r$ is negative and it is better to write this expression in the form

$$S = \frac{a(-r^n + 1)}{-r + 1} = \frac{-a(r^n - 1)}{-(r - 1)},$$

or,

$$S = \frac{a(r^n - 1)}{r - 1}. \tag{36}$$

Formula (36) gives the sum of the terms of an increasing geometrical progression and (35) of a decreasing progression.

The indicated sum of a sequence of numbers formed and related by some specified law or rule, such as an arithmetical or geometrical progression, is called a *series*. Thus, the equation or sum (34) is a series. Another example of a series is the binomial series of Chapter 9.

119. Application of the Formulas. By means of formulas (33) and (35) or (36), any one of the five numbers a, r, n, l, S can be expressed in terms of any other three. Thus (33) gives l in terms of a, r and n; (35) or (36) gives S in terms of a, r and n; by substituting the value of a from (33) in (35) S would be given in terms of l, r and n; etc.

Suppose a, r and l are known and it is desired to find n. In this case the exponential equation (33) is to be solved for n. Dividing (33) by a we get

$$r^{n-1} = \frac{l}{a}.$$

$$\therefore \quad \log (r^{n-1}) = \log \left(\frac{l}{a}\right).$$

Applying the law for the logarithm of a power on the left and for the logarithm of a quotient on the right, this gives,

$$(n - 1) \log r = \log l - \log a \tag{37}$$

$$n - 1 = \frac{\log l - \log a}{\log r}.$$

$$\therefore \quad n = \frac{\log l - \log a}{\log r} + 1. \tag{38}$$

This formula gives n in terms of a, r and l. Similarly (35) or (36) may be solved for n in terms of a, r and S.

If, in (33), a, l and n are known r may be found as follows: From (33) we have already derived (37), and from (37),

$$\log r = \frac{\log l - \log a}{n - 1}. \tag{39}$$

With a, l and n known the right member can be calculated, and this

gives the value of $\log r$. With $\log r$ thus known, r is read from the table as the anti-logarithm. This formula, therefore, expresses r in terms of a, l and n. As thus found r will be a decimal number. Similarly, (35) or (36) may be solved for r in terms of a, S and n, etc.

Since the number of terms in a progression cannot be fractional, n as found from (38) must always be a whole number. If the logarithm tables used do not have enough figures to give n exactly, the nearest whole number to the value found is to be used.

Let us find the sum of the terms of the geometrical progression whose last term was calculated in article 118. We had $a = 3\frac{1}{2}$, $r = 2$ and $n = 13$. Since r is greater than 1, formula (36) is used. This gives

$$\begin{aligned} S &= \frac{3\frac{1}{2}(2^{13} - 1)}{2 - 1} = 3\tfrac{1}{2}(8192 - 1) \\ &= 28{,}668.5 \end{aligned}$$

The common ratio of a geometrical progression of 6 terms is 3 and the sum of the terms is 728. Find the first and last terms and write out the sequence.

Here $r = 3$, $n = 6$, $S = 728$, and a, l are unknown. Substituting the given values in formulas (33) and (36) we have the two equations

$$l = a \cdot 3^5, \quad 728 = \frac{a(3^6 - 1)}{3 - 1},$$

to be solved for a and l. Simplifying, these become

$$l = 243a, \quad 728 = \frac{a(729 - 1)}{2} = 364a.$$

From the second,

$$a = \frac{728}{364} = 2,$$

and this in the first gives

$$l = 243 \times 2 = 486.$$

The progression is, therefore,

$$2, \quad 6, \quad 18, \quad 54, \quad 162, \quad 486.$$

The first term of a geometrical progression of 8 terms is 2 and the last is 4374. What is the common ratio?

Here $a = 2$, $l = 4374$ and $n = 8$. Formula (39), therefore, gives

$$\log r = \frac{\log 4374 - \log 2}{8 - 1}$$

$$= \frac{3.6409 - 0.3010}{7} = \frac{3.3399}{7}$$

or,

$$\log r = 0.4771.$$

$$\therefore \quad r = 3.$$

The first term of a geometrical progression with a common ratio of $1\frac{1}{2}$ is 4 and the last term is $20\frac{1}{4}$. Find the number of terms.

Here $a = 4$, $l = 20.25$ and $r = 1.5$. According to formula (38), therefore,

$$n = \frac{\log 20.25 - \log 4}{\log 1.5} + 1.$$

$$= \frac{1.3064 - 0.6021}{0.1761} + 1 = \frac{.7043}{.1761} + 1.$$

$$\therefore \quad n = 4 + 1 = 5.$$

120. Power Series. The sum of a geometrical series is given by (34), namely,

$$S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}, \tag{34}$$

where n is the number of terms. If n is specified as a finite number, the expression on the right is a polynomial of n terms, but if n is not specified, there is an indefinite number of terms, and (34) becomes a *series.* In this case it is written

$$S = a + ar + ar^2 + ar^3 + ar^4 + \ldots. \tag{40}$$

If the number of terms is infinitely great (n equals "infinity") the expression on the right of (40) is called an *infinite series.*

We have already had one example of a series, namely, the binomial theorem expansion. The form of the separate terms in the binomial series is quite different in general, however, from the form of the terms in (40).

In the series (40) the different powers of r all have the same coefficient, a. If the coefficients are all different and are denoted by a_0, a_1, a_2, a_3, etc., (40) becomes

$$S = a_0 + a_1r + a_2r^2 + a_3r^3 + a_4r^4 + \ldots. \tag{41}$$

This is called a *power series* and the variable r is called the variable

of the series. The series is referred to as "power series in r" and S is said to be the *sum to infinity*. Power series are of the greatest importance in higher pure and applied mathematics.

If, in (40) or (41), r is greater than 1, each term is greater than the preceding term and the sum S has no definite limiting value, but increases indefinitely as the number of terms increases. If, on the other hand, r is less than 1 the successive terms are smaller and smaller, and as the number of terms n becomes infinite the last term approaches zero in value. In this case, the sum S approaches a definite limiting value as n becomes infinite and is said to be the "limit of the sum of the series as n approaches infinity."

By taking out a as a common monomial factor of the terms of the infinite series (40), it can be written

$$S = a(1 + r + r^2 + r^3 + r^4 + \ldots). \tag{42}$$

The formula (35) for S gives also (with r less than 1),

$$S = \frac{a(1 - r^n)}{1 - r}. \tag{43}$$

Now, when r is less than 1, it is a fraction and the product of a fraction by itself is a still smaller fraction, that is, it is a fraction of a fraction. Therefore, as n increases r^n decreases and in the limit, when n is infinite, r^n is zero. In this case, the expression $1 - r^n$ becomes equal to 1 and (43) becomes

$$S = \frac{a}{1 - r} = a\left(\frac{1}{1 - r}\right). \tag{44}$$

Since S in both (42) and (44) represents the same sum, then

$$a\left(\frac{1}{1 - r}\right) = a(1 + r + r^2 + r^3 + r^4 + \ldots).$$

$$\therefore \quad \frac{1}{1 - r} = 1 + r + r^2 + r^3 + r^4 + \ldots. \tag{45}$$

Although this expression is derived from an infinite decreasing geometrical series, it may be verified by carrying out the division $1 \div (1 - r)$ by ordinary algebraic long division, the quotient never coming out exact.

The expression $\dfrac{1}{1 - r}$ can also be written as $(1 - r)^{-1}$ and we then

have

$$(1 - r)^{-1} = 1 + r + r^2 + r^3 + r^4 + \dots. \tag{46}$$

In this case, the expansion of $(1 - r)^{-1}$ can be verified by applying the binomial theorem, and we have the result that an infinite geometrical series is a special case of the *binomial series.*

121. Construction of Logarithm Tables. It is stated in article 95 that the mantissas of logarithms are calculated by special methods which are explained in the calculus. The calculus is required for a full derivation and discussion of the methods, but an outline of these methods can be given by the use of the principles and procedures developed in this chapter. Logarithms are calculated by means of infinite power series, and the series used are developed from the binomial theorem and the *binomial series.*

When the number $e = 2.71828+$, referred to in article 93 as the base of the system of *natural logarithms*, is properly defined, it is shown in calculus or in higher algebra that the value of any power of e with exponent y is expressed by the infinite power series

$$e^y = 1 + y + \frac{y^2}{\underline{|2}} + \frac{y^3}{\underline{|3}} + \frac{y^4}{\underline{|4}} + \frac{y^5}{\underline{|5}} + \dots, \tag{47}$$

where $\underline{|5} = 1 \times 2 \times 3 \times 4 \times 5 = 120$ is called the "factorial product" of the number 5, or simply "factorial 5," and similarly for any other number, as defined in article 52. The series (47) is called the "exponential series," and when $e^y - 1 = x$ it is shown by the calculus or by the process called *reversion of series* in higher algebra that $y = \log_e (1 + x)$ and

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots. \tag{48}$$

This series is called the "logarithmic series" and from it we can now easily obtain the special infinite series used to construct tables of logarithms.

If in series (48) we replace x with $-x$ the series becomes

$$\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots. \tag{49}$$

Now subtract series (49) from (48), member by member and term by term. The result is

$$\log_e (1 + x) - \log_e (1 - x) = 2x + 2 \cdot \frac{x^3}{3} + 2 \cdot \frac{x^5}{5} + \ldots,$$

and by making use of formula (5) of article 87 and also taking out the common factor 2 in the series, this may be written

$$\log_e \left(\frac{1 + x}{1 - x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right]. \tag{50}$$

By giving x any fractional value at will the fraction $\left(\frac{1 + x}{1 - x}\right)$ can be made to equal any desired number, and the logarithm of this number is then equal to the sum "to infinity" of the series on the right of (50), found by giving x the value used in the fraction on the left of the equal sign in (50). By using a sufficient number of the terms in the series the value of the logarithm may be calculated to any desired number of decimal places. Logarithms obtained by this series are natural logarithms (to base e) as shown by the formula, and common logarithms (base 10) are then found from the natural logarithms by the formula (14) of article 93.

When x is a fraction (common or decimal) in series (50) each term in the series is slightly less in value than the preceding term and the series is said to *converge*. The terms decrease in value, or the series converges, so slowly, however, that a very great number of terms is needed to obtain the value of the logarithm to any useful number of decimal places. A series which converges more rapidly is derived from (50) as follows:

Let n be such a number that the fraction

$$\frac{1 + x}{1 - x} = \frac{n + 1}{n}, \quad \text{or, solving this,} \quad x = \frac{1}{2n + 1}.$$

Substituting this value of the fraction in terms of n on the left of (50) and the value of x in terms of n in the series on the right, series (50) becomes

$$\log_e \left(\frac{n + 1}{n}\right)$$
$$= 2\left[\frac{1}{(2n + 1)} + \frac{1}{3(2n + 1)^3} + \frac{1}{5(2n + 1)^5} + \frac{1}{7(2n + 1)^7} + \ldots\right].$$

Now $\log_e \left(\frac{n + 1}{1}\right) = \log_e (n + 1) - \log_e n$. Using this result on the

left, and transposing then to the right the term $-\log_e n$, we get finally

$$\log_e (n + 1)$$

$$= \log_e n + 2\left[\frac{1}{(2n+1)} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right] \quad (51)$$

which is the series used to calculate natural logarithms for the construction of tables. Common logarithms are obtained from the natural logarithms as explained before.

In series (51) the fractions in the series all have the numerator 1 and the denominators, being multiples of powers of the number $2n + 1$, increase very rapidly from term to term, so that the series converges very rapidly and not many terms are needed to calculate the natural logarithm of any number to as many decimal places as desired.

The series (51) is used as follows: When the value of the natural logarithm of any number n ($\log_e n$) is known this logarithm is added to 2 times the value of the series calculated with the number n in the fractions, and the result is then the natural logarithm of the next number, $n + 1$. In order to use the series, therefore, it is necessary to know the natural logarithm of some number in advance. But we have seen that the logarithm of the number 1 is zero in any system, i.e., $\log_e 1 = 0$. Using this value of $\log_e 1$ in the series and using $2n + 1 = 3$ in each of the fractions, the series gives the value of $\log_e (1 + 1) = \log_e 2$. With the value of log 2 now known the logarithm of the next number, log 3 is found by using $n = 2$. With log 2 now known, $\log 4 = \log (2^2) = 2(\log 2)$ is found by formula (6), article 88. Then log 5 is found as $\log (4 + 1)$ with $n = 4$. Next, $6 = 3 \times 2$ and therefore $\log 6 = \log 3 + \log 2$. In this way the natural logarithm of every whole number from 1 on as far as desired is found, and those of decimal fractions are then found by the rules applying to characteristics and decimal points.

As an illustration of the use of the series (51) we compute here the logarithm of the number 2. Using $n = 1$, then $\log_e n = \log_e 1 = 0$, $2n + 1 = 3$, and the series (51) becomes

$$\log_e 2 = 0 + 2\left[\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \frac{1}{9.3^9} + \ldots\right]$$

or,

$$\log_e 2 = 2\left[\frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \frac{1}{15{,}309} + \frac{1}{777{,}147} + \ldots\right];$$

and on working out the decimal values of the fractions in the series and adding the terms,

$$\begin{aligned}\log_e 2 \\ &= 2(.333333 + .012346 + .000823 + .000065 + .000006 + \ldots) \\ &= 2 \times .346573 = .693146 = 0.69315,\end{aligned}$$

to five places. From this, by formula (14) of article 93,

$$\log_{10} 2 = 0.43429 \times 0.69315 = 0.30103 = 0.3010,$$

to four places, as given in the table at the end of this book.

Chapter 22

PERMUTATIONS, COMBINATIONS AND PROBABILITY

122. Introduction. The subjects of permutations, combinations and probability are very closely related and are of great interest and usefulness in connection with card games, games of chance, life insurance, inspection sampling, scientific measurements, statistics and in any activity involving large numbers of things or objects or many repetitions of certain definite operations. These subjects, and that of probability in particular, had their origin in the study of games of chance, but when it was found that the principles involved are susceptible of precise mathematical formulation, they began to be studied for their own interest as branches of mathematics.

From these beginnings have grown the several branches of pure and applied mathematics known as combinations, probability, statistics, average and mean values, method of least squares, theory of errors, statistical mechanics. Various ones of these subjects are of the greatest importance in the highly technical sciences of insurance, engineering, physics and chemistry, as well as others, and all are treated more or less completely in books devoted to the individual subjects or in certain chapters of mathematical books devoted to related subjects.

In this chapter only a brief introduction to the subject will be given, including definitions of the terms used, descriptions of some of the methods of calculation, and development of some of the chief algebraic relations and principles.

123. Combination and Permutation. Let us consider a group of n objects or things represented by the n letters a, b, c, d, etc. Any set consisting of the number r of these letters or objects, taken *without regard to their order*, is called a *combination* of the n letters r at a time, or more briefly, an *r-combination of the n* letters. The number of such combinations which can be made is denoted by the symbol C_r^n. In this symbol, n and r are not exponents, but are called the *upper* and *lower index* of C.

As an illustration, the 2-combinations of the 4 letters a, b, c, d are ab, ac, ad, bc, bd, cd, and there are 6 of these. That is, $n = 4$, $r = 2$ and $C_2^4 = 6$.

On the other hand, any arrangement of r of these n objects or letters in a row *in a definite order* is called a *permutation* of the n letters r at a time, or more briefly, an *r-permutation of the n* letters. The number of such permutations is denoted by the symbol P_r^n.

As an illustration, the 2-permutations of the 4 letters a, b, c, d are

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd,$$

$$ba, \quad ca, \quad da, \quad cb, \quad db, \quad dc,$$

and there are 12 of these. That is $P_2^4 = 12$. It is to be noted that ab and ba are the same combination but are different permutations.

Unless otherwise specified, it is to be understood in what follows that the n objects a, b, c, d, . . . are all different and that no letter is repeated in a combination or a permutation. Also, letters will always be understood to represent any objects.

We next give a preliminary discussion of permutations because of their importance in connection with combinations.

124. Number of r-Permutations of n Objects. Suppose there are m ways of doing a certain thing, and that for each way of doing it there are n ways of doing a second thing. There are, then, $m \times n$ ways of doing both together. Similarly if one thing can be done in m ways, and for each of these the second can be done in n ways, and for each of the $m \cdot n$ ways of doing the two a third can be done in p ways, etc., the entire number of ways in which all the things can be done is $m \cdot n \cdot p \ldots$.

For example, how many three-figure numbers can be formed from the nine digits (figures) 1, 2, 3, 4, 5, 6, 7, 8, 9?

Any one of the *nine* figures may be chosen for the first figure of any one of the numbers. When this one is chosen any one of the remaining *eight* can be chosen for the second figure; and then any one of the remaining *seven* for the third figure. The choices may therefore be made in $9 \times 8 \times 7$ ways and there are 504 numbers of the kind required. As stated above, no figure is used twice in any one of these numbers.

Similarly there are $9 \times 8 \times 7 \times 6 = 3024$ four-figure numbers, $9 \times 8 \times 7 \times 6 \times 5$ five-figure numbers, etc. In general, the number of r-figure numbers which can be formed of the 9 figures is $9 \cdot 8 \cdot 7 \cdot 6 \cdot \ldots$

to r factors. This may be written $9(9-1)(9-2)(9-3)\ldots$ and in the same way the number of r-figure numbers which can be formed from n digits is $n(n-1)(n-2)(n-3)\ldots$ to r factors.

Now, each of these numbers formed consists of r figures arranged in a definite order, and, if the order is changed, the number is changed. Each one, therefore, forms an r-permutation of the n figures. The total number of such permutations is, therefore,

$$P_r^n = n(n-1)(n-2)(n-3)\ldots \text{ to } r \text{ factors.} \tag{52}$$

Thus, the number of permutations of 5 letters taken 1, 2, 3, 4 or 5 at a time is

$P_1^5 = 5$, (1 factor),

$P_2^5 = 5(5-1) = 5 \times 4 = 20$, (2 factors),

$P_3^5 = 5(5-1)(5-2) = 5\cdot 4\cdot 3 = 60$, (3 factors),

$P_4^5 = 5(5-1)(5-2)(5-3) = 5\cdot 4\cdot 3\cdot 2 = 120$, (4 factors),

$P_5^5 = 5(5-1)(5-2)(5-3)(5-4) = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$, (5 factors).

In these, the second factor is $5-1$ or $5-(2-1)$, the third factor is $5-2$ or $5-(3-1)$, the fourth is $5-(4-1)$, and, in general, the rth factor is $5-(r-1)$, where the total number of figures is 5. Therefore, when there are n letters the rth factor is $n-(r-1) = (n-r+1)$. The complete product in (47) is, therefore,

$$P_r^n = n(n-1)(n-2)(n-3)\ldots(n-r+1), \tag{53}$$

and this is the total possible number of r-permutations of n objects.

For example, the total number of 6-figure numbers which can be written with the 9 digits is (since $n-r+1 = 9-6+1 = 4$),

$$P = 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 60{,}480.$$

If all the n objects are used in each permutation and only the order is changed, then $r = n$, the last factor in (53) is $n-n+1$, and

$$P_n^n = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1,$$

which is the continued product of all the numbers from n to 1, or 1 to n. This *continued product of all numbers from* 1 *to* n is called *factorial* n and is represented by the symbol $\underline{|n}$ or $n!$ (see article 52). Therefore,

$$P_n^n = \underline{|n}. \tag{54}$$

Formulas (53) and (54) are derived on the assumption that no letter

or object is used twice in any permutation. If the letters are repeated in an r-permutation, any one of the set of n letters may be chosen as the first letter, and again any one of the entire set for the second, for the third, and so on. Therefore, each of the r factors in (53) is n and we have the result that $P_r^n = n \cdot n \cdot n \cdot n \ldots$ to r factors.

$$P_r^n = n^r \textit{ with repetitions.} \tag{55}$$

Here n^r has its ordinary meaning, the rth power of n.

An important case is that in which some of the objects are alike and all are used in each permutation. Consider the 5 letters a, a, a, b, c, of which 3 are alike. Let us refer to the permutations of these letters as (i), and compare them with the permutations (ii) of a, a', a'', b, c, all of which are different, to which formula (53) applies. If we take any one of the permutations (i) as *abaca* and interchange the a's without disturbing b and c we get nothing new. But if we treat the corresponding permutation of (ii), $a''ba'ca$, in the same way we obtain, according to (54), $\lfloor 3$ distinct permutations. Hence, to each permutation of (i) corresponds $\lfloor 3$ permutations of (ii). The number of permutations (i) is $\lfloor 5$ and, therefore, the total number of distinguishable permutations of both kinds is $\lfloor 5 \div \lfloor 3$ or $\dfrac{\lfloor 5}{\lfloor 3}$.

Similarly if there are 2 others alike in the five letters there will be a total of $\left(\dfrac{\lfloor 5}{\lfloor 3}\right) \div \lfloor 2$ or $\dfrac{\lfloor 5}{\lfloor 3 \times \lfloor 2}$, and, in general, if there are in n objects, p alike, q others alike, s others alike, etc., the total number of different permutations is

$$N = \frac{\lfloor n}{\lfloor p \cdot \lfloor q \cdot \lfloor s \ldots}. \tag{56}$$

This formula is of great importance in the kinetic theory of gases in physics and chemistry, where a gas is found to be composed of a very large but calculable number of particles called molecules.

125. Number of r-Combinations of n Objects. If all the r-combinations (not permutations) of n objects are formed in accordance with the definition and the illustration of article 123, and the objects of each combination are then arranged in all possible different orders in turn, the result will be that all the r-permutations are obtained.

In this process, all the r-permutations of the r objects in each com-

bination are formed, and according to formula (54) this is $\underline{|r}$. The total number of r-permutations of all the n objects is, therefore, the number of r-combinations multiplied by $\underline{|r}$. That is,

$$\underline{|r} \times C_r^n = P_r^n.$$

Dividing by $\underline{|r}$,

$$C_r^n = \frac{P_r^n}{\underline{|r}}.$$

Substituting in this the value of P_r^n given by formula (53) we have

$$C_r^n = \frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)}{\underline{|r}}. \quad (57)$$

This formula gives the number of combinations (order disregarded) of n different objects taken r at a time.

For example, how many different 5-card hands are there in a deck of 52 cards?

Here $n = 52$, $r = 5$, $n - r + 1 = 48$, and formula (57) gives

$$C_5^{52} = \frac{52 \times 51 \times 50 \times 49 \times 48}{\underline{|5}} = 2{,}598{,}960.$$

There are, therefore, over $2\frac{1}{2}$ million different 5-card hands in a 52-card deck.

If the binomial series (25) of article 52 be examined, it will be seen that the coefficient of the third term is $\dfrac{n(n-1)}{\underline{|2}}$; that of the 4th is $\dfrac{n(n-1)(n-2)}{\underline{|3}}$; that of the 5th is $\dfrac{n(n-1)(n-2)(n-3)}{\underline{|4}}$; or $\dfrac{n(n-1)(n-2)(n-4+1)}{\underline{|4}}$. Similarly, the coefficient of the $(r+1)$th term is $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}$. But this is the right side of formula (57). We have, therefore, the result that

The coefficient of the $(r+1)$th term of the nth power of a binomial equals the number of r-combinations of n different objects.

The binomial expansion of $(a+b)^n$ can, therefore, be written

$$(a+b)^n = a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \ldots + C_n^n b^n. \quad (58)$$

126. Total Number of all r-Combinations of n Objects. Suppose

all the combinations of n objects taken one at a time (the number of the objects) are found, and then all the combinations taken two at a time, then all those three at a time, and so on until all those taken n at a time (all at once) are formed How many combinations will there be altogether?

To begin with, it is obviously the sum of the 1-combinations, the 2-combinations, the 3-combinations, etc. That is, if C denote the total number,

$$C = C_1^n + C_2^n + C_3^n + \ldots + C_n^n. \tag{59}$$

In the binomial expansion (58), let us put $a = 1$, $b = 1$. Then $(a + b)^n = (1 + 1)^n = 2^n$, and (58) becomes

$$2^n = 1 + C_1^n + C_2^n + C_3^n + \ldots + C_n^n.$$

Transpose,

$$2^n - 1 = C_1^n + C_2^n + C_3^n + \ldots + C_n^n.$$

Comparing this result with (59) it is seen at once that

$$C = 2^n - 1. \tag{60}$$

That is, the total number of ways in which one or more things may be chosen from n things is $2^n - 1$. For example, the total number of 5 things taken 1, 2, 3, 4, or 5 at a time is

$$C = 2^5 - 1 = 32 - 1 = 31.$$

127. Combinations with Repetitions. Let us consider the number of ways in which three of the four digits 1, 2, 3, 4, can be selected when repetitions are allowed. An example of such selections in which all three selected digits are the same is 111, one in which two are the same is 112, and one in which none are the same is 124.

If to the three digits in each of these combinations 0, 1, 2, respectively, be added, the combinations become 123, 124, 136, and these are 3-combinations of the 4 digits 1, 2, 3, 4, *without* repetitions. Similarly, if we make out all the combinations like 111, 112, 124 such that no digit is followed by a lower one, and then add 0, 1, 2, respectively to the three digits in each combination, we shall obtain once, and only once, every one of the 3-combinations without repetitions of the $4 + (3 - 1)$ or 6 digits 1, 2, 3, 4, 5, 6. The number of these 3-combinations is C_3^6, and this is the same as the number of combinations of 1, 2, 3, 4, *with* repetitions.

In the same way as for the $4 + (3 - 1)$ 3-combinations of 4 digits,

we have $n + (r - 1)$ r-combinations of n digits or any other things with repetitions. Stated more fully:

The number of r-combinations of n different things with repetitions is the same as the number of r-combinations of $(n + r - 1)$ *different things without repetition, that is,* C_r^{n+r-1}.

Comparing the r-combinations C_r^{n+r-1} with the r-combinations C_r^n it is seen that instead of n we now have $n + r - 1$. If, therefore, we replace n by $(n + r - 1)$, formula (57) will give the number of r-combinations of n things *with repetitions.* Doing this we have in the numerator on the right of (57),

$$(n + r - 1)[(n + r - 1) - 1][(n + r - 1) - 2] \ldots [(n + r - 1) - r + 1],$$

or,

$$(n + r - 1)(n + r - 2)(n + r - 3) \ldots n.$$

Now, the factors in this product decrease from left to right, each factor being 1 less than the preceding factor, from $n + r - 1$ on down to n. From right to left, therefore, they *increase* by 1 from n on up to $n + r - 1$. Since the order of the factors is immaterial in multiplication, we may write the product as

$$n(n + 1)(n + 2)(n + 3) \ldots (n + r - 1).$$

This is the final form of the *numerator* in formula (57) when repetitions are allowed. The formula, then, becomes,

$$C_r^n = \frac{n(n + 1)(n + 2)(n + 3) \ldots [n + (r - 1)]}{\underline{|r}} \text{ with repetitions.} \qquad (61)$$

This formula should not be confused with (57), which gives the number of r-combinations of n different things *without* repetition, but the two formulas should be compared.

As an example of the use of (61), how many different throws can be made with four dice?

As any one of the faces marked 1, 2, 3, 4, 5 or 6 may turn up on any 1, 2, 3 or 4 of the dice, the number of possible throws is the number of 4-combinations with repetitions, of the 6 faces. In (61), therefore, we have $n = 6$, $r = 4$, $r - 1 = 3$ and the formula gives

$$C_4^6 = \frac{6(6 + 1)(6 + 2)(6 + 3)}{\underline{|4}} = \frac{6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4} = 126.$$

128. Probability. Consider any event which, if given an opportunity to happen, must happen or fail to happen in only one of a

limited number of ways *all equally likely*. All the equally likely ways in which such an event can happen or fail are called *possible cases* with respect to the event. All the ways in which it can happen are called *favorable cases*, and all the ways in which it can fail *unfavorable cases*. An opportunity to happen is called a *trial* of the event. Using these terms we have the following

DEFINITION: *The probability or chance of an event is the ratio of the number of favorable cases to the total number of possible cases.*

Formulating this definition in symbols, if n denote the number of possible cases, a the number of favorable and b the number of unfavorable cases, the probability is

$$p = \frac{a}{n} = \frac{a}{a + b}. \tag{62}$$

As an example, consider the turning up of the ace in throwing a die. One face must turn up, and there is no reason for expecting any one to turn up rather than another, that is, all are equally likely. The total number of faces is the total number of possible cases, $n = 6$. The number of favorable cases is the number of aces, $a = 1$; the number of unfavorable cases is then $b = 5$; and the probability is $p = \frac{1}{6}$. This is read "one sixth," "one to six," or "one in six."

As another example, the chance of drawing a white ball from a bag known to contain three white and two black balls is $p = \dfrac{a}{a + b} = \dfrac{3}{3 + 2} = \dfrac{3}{5}$. Similarly the probability or chance of drawing an ace at one draw from a deck of 52 cards is $4/52 = 1/13$.

129. Some Principles of Probability. If an event is certain to happen, the ways in which it can fail are zero, that is, $b = 0$ and $a = n$. In this case the probability is $p = a/n = n/n = 1$. If the event is certain to fail there are no ways in which it can happen and $a = 0$. The probability is then $p = 0/n = 0$. In every other case a is greater than zero and less than 1 and $p = a/n$ is a positive fraction (between zero and 1).

These results are summed up by saying that

(A) *If an event is certain to happen, its probability is* 1; *if it is certain to fail, its probability is* 0; *in all other cases, its probability is between* 0 *and* 1.

Since a of the n possible cases of an event favor the occurrence of the event under consideration, then $b = n - a$ cases favor its non-

occurrence. By definition, therefore, the probability of its non-occurrence is $(n - a)/n$. But $\frac{n-a}{n} = \frac{n}{n} - \frac{a}{n} = 1 - \frac{a}{n}$, and $\frac{a}{n} = p$, the probability of its occurrence. The probability of its non-occurrence is, therefore, $1 - p$. Concisely expressed,

(B) *If the probability that an event will happen is p, the probability that it will not happen is $1 - p$.*

If the number of favorable cases a with respect to an event is greater than the number of unfavorable cases b, $a > b$, it is said that the *odds* are in favor of the event. If $b > a$ the odds are said to be against it. If $a = b$ the odds are even.

Now if $a > b$, $(a + a) > (b + a)$, or $(a + b) < 2a$ and hence $p = \frac{a}{a+b} > \frac{a}{2a}$ or $\frac{1}{2}$. That is, $p > \frac{1}{2}$. If $b > a$, $(a + b) > 2a$ and $p = \frac{a}{a+b} < \frac{a}{2a}$ or $\frac{1}{2}$. That is, $p < \frac{1}{2}$. Finally when $a = b$, $a + b = 2a$ and $p = \frac{a}{a+b} = \frac{a}{2a} = \frac{1}{2}$. Therefore.

(C) *When the odds are in favor of an event its probability is greater than $\frac{1}{2}$, when the odds are against it the probability is less than $\frac{1}{2}$, and when the odds are even the probability equals $\frac{1}{2}$.*

130. Different Meanings of Probability. In the different uses and applications of the theory of probability, such as those mentioned in article 122, somewhat different meanings are attached to the term. Some of these will be indicated here.

The fraction a/n, which is defined as the probability of an event, has very little significance so far as the outcome of a *single trial*, or a small number of trials, is concerned. It indicates rather the frequency with which the event would occur in the course of a very large number of trials.

Thus, if a die is thrown a thousand times, let us say, it will be found that the ratio of the number of aces thrown to the total number of throws will very closely approach 1/6. That is, of the thousand throws, approximately 167 will be aces.

The definition (62) applies only in cases in which it is possible to count the total number of ways in which an event can happen. There are many classes of events in which this is not possible. For instance, in considering the duration of the life of a single person, it is impossible to enumerate all the equally likely ways in which, or reasons why, the

person may die at a certain age or live on past that age. It is possible, however, to determine the number of persons per thousand or per hundred thousand who die at a certain age, that is, the number of times the event (death) happens out of the total possible number. In this case, the ratio of the number of deaths at a certain age, to the total number of people, living and dead, of that age, is the probability of the event, that is, of death at that age. This probability indicates the *frequency* with which events of such a class may reasonably be expected to happen in the case of a very great number of *future* trials.

Thus, if it should be learned from census reports that of each 100,000 persons of age sixty in 1930 about two-thirds were still living in 1940, it would be said that the probability that a person of age sixty in 1940 would be alive in 1950 is 2/3. The apparent chances of his being alive are, then, two to three.

The fraction $p = a/n$ is also used to indicate the strength of *expectation* that an event will occur on a *single* trial. The greater the ratio of the number of favorable cases to the number of possible cases, or the greater the frequency with which similar events have happened in the past, the stronger is the expectation that the particular event under consideration will occur on the single trial under consideration.

Thus, before a football game it may be said that the odds of A's winning over B are 3 to 2. This means, as in article 128, that the possibilities favorable to A are $a = 3$ to $b = 2$ possibilities favorable to B, and the probability of A's winning the future game under consideration is $p = \dfrac{a}{a+b} = \dfrac{3}{3+2} = \dfrac{3}{5}$. This means that the expectation of A's victory is about as strong as the expectation of drawing a white ball from a box known to contain five balls, three of which are white.

131. Probability of Independent and Exclusive Events. If the occurrence or non-occurrence of any one of a number of events is not affected in any way by that of any of the others, the events are said to be *independent* events.

Consider two independent events whose separate probabilities, according to (62), are $p_1 = a_1/n_1$ and $p_2 = a_2/n_2$. The number of equally likely possible cases for and against the first is therefore n_1 and for and against the second n_2, and since the events are independent, any one of the n_1 cases may occur with any one of the n_2 cases. Hence the number of the equally likely possible cases for and against the

occurrence of both is $n_1 \times n_2$. Similarly, the number of cases favorable to the occurrence of both is $a_1 \times a_2$. Therefore, the probability that both will occur is, by definition,

$$p' = \frac{(a_1a_2)}{(n_1n_2)} = \frac{a_1}{n_1} \cdot \frac{a_2}{n_2} = p_1 \cdot p_2.$$

Similarly, considering the probability p' of these two events and the probability p_3 of a third independent event, the probability of all three occurring is $p = p' \cdot p_3 = p_1 \cdot p_2 \cdot p_3$, and so on. Therefore,

(D) *The probability that all of a set of independent events will occur is the product of their separate probabilities.*

It is to be noted that the combined probability is less than any one of the several, for all are fractions and the product of several fractions is a fraction less than any one of the fractions.

If several events are so related that only one of them can occur, they are said to be mutually *exclusive.*

Consider two mutually exclusive events A and B. The possible cases with respect to *both*, all mutually exclusive, are: (i) A happens, B fails; (ii) A fails, B happens; (iii) A fails, B fails (by definition both cannot occur). Let the numbers of equally likely possible cases of these three kinds be l, m, n, respectively. Then the total number of cases is $l + m + n$, and by (62) the chance that either A *or* B happens is

$$\frac{l + m}{l + m + n};$$

the chance that A happens is

$$\frac{l}{l + (m + n)};$$

the chance that B happens is

$$\frac{m}{m + (l + n)}.$$

But by addition of fractions

$$\frac{l + m}{l + m + n} = \frac{l}{l + (m + n)} + \frac{m}{m + (l + n)}.$$

Therefore, the chance that *either* A *or* B will happen is the sum of the single chances of A and B.

Considering A and B together, and a separate third exclusive event

C, it is seen that this result applies to the three, and so on for any number. Therefore,

(E) *The probability that some one or other of a set of mutually exclusive events will occur is the sum of their separate probabilities.*

The principles established in this article will be useful in connection with the considerations of the following article.

132. Repeated Trials of a Single Event. An interesting and important question in the theory of probability is that of the chance that a certain event will occur a specified number of times in a series of trials when the chance or expectation of its occurrence on a single trial is known.

If a certain event be given a total of n trials and we consider any set of r of these trials the number of the remaining trials is $n - r$. If the probability of its occurrence on a single trial is p, then, since the resultant probability of its occurrence on all of the r trials is the same as the probability of r different single independent events, this resultant probability is, according to the principle (D), article 131, $p \cdot p \cdot p \cdot p \ldots$ to r factors, that is, p^r. If q is the probability that it will fail on each of the remaining $n - r$ trials, then by the same principle the probability that it will fail on all of them is q^{n-r}. Then by the same principle, the total probability that it will happen on all the r trials and fail on all the $n - r$ trials, is the product $p^r \cdot q^{n-r}$.

But, according to the principle (B), the probability q of failure on any one trial is $1 - p$, p being the probability of occurrence on one trial. Substituting $q = 1 - p$ in the product just obtained, we have for the probability p' of its occurrence on the r and failing on the $n - r$ trials

$$p' = p^r(1 - p)^{n-r}. \tag{63}$$

Now, since there are n trials altogether the set of r trials may be selected from the n in a number of different r-combinations equal to C_r^n and each of these is mutually exclusive. Then, by (E) the total probability is the sum of the probabilities of these C_r^n sets of r trials, and since the probability of each set is p' the total probability P is

$$P = C_r^n \times p'.$$

Hence by (63), we have, finally,

$$P = C_r^n p^r (1 - p)^{n-r}. \tag{64}$$

Stated in words, this says that,

(F) *If the probability that an event will occur on a single trial is p, the probability that it will occur exactly r times in n trials is*

$$C_r^n \cdot p^r(1 - p)^{n-r}.$$

This r-probability of n trials of probability p each may be expressed in terms of p, r, and n by means of formula (57). Substituting the value of C_r^n there given, in (64) we have

$$P = \frac{np^r}{\underline{|r}}(1 - p)^{n-r}(n - 1)(n - 2)(n - 3) \ldots (n - r + 1). \quad (65)$$

In the formula (63) it is to be noted that the r-probability of n trials is, according to (58), the term containing p^r in the expansion of $(p + q)^n$ by the binomial theorem, for $C_r^n \cdot p^r(1 - p)^{n-r} = C_r^n \cdot p^r q^{n-r}$, since $1 - p = q$.

As an example, what is the chance that the ace will turn up exactly twice in five throws of a die, or that two aces will turn up if five dice are thrown at once?

Here $n = 5$, $r = 2$ and the probability that the one face in six will turn up on any one throw is $p = \frac{1}{6}$. Hence $n - r = 3$, $n - r + 1 = 4$ and by formula (65) the probability of exactly two aces, no more and no less, in five throws is

$$P = \frac{5(\frac{1}{6})^2}{2}(1 - \tfrac{1}{6})^3 \times 4 = \frac{5(\frac{1}{36})}{2} \times (\tfrac{5}{6})^3 \times 4$$

$$= \tfrac{10}{36} \times \tfrac{125}{216} = \tfrac{625}{3888}.$$

$$\therefore \quad P = 1 : 6\tfrac{1}{4} \text{ (approximately).}$$

That is, the chance of just two aces in five throws (or two aces in five dice thrown at once) is 1 in $6\frac{1}{4}$, or the odds against it are $6\frac{1}{4}$ to 1.

The principle (F), or formula (64), gives the probability that an event will occur *exactly* r times in n trials. Let us now consider the probability of its occurring *not less than* r times in n trials (possibly more).

If it occurs any number of times greater than r, it will then of course occur not less than r times. But any number from r to n is n, $n - 1$, $n - 2$, $n - 3$, . . . or so on, down to r. The probability of its occurrence on all of the n trials is, according to (D), p^n. Let this probability be denoted by p_1. Then,

$$p_1 = p^n. \quad (66)$$

The probability of its occurrence on $n - 1$ of the n trials is, by (64), $C^n_{n-1}p^{n-1}(1 - p)^{n-(n-1)}$. But, since $1 - p = q$ as in (63), and $n - (n - 1) = 1$, this probability, which is denoted by p_2, is

$$p_2 = C^n_{n-1}p^{n-1}q. \tag{67}$$

Similarly, for $n - 2$ occurrences in n trials,

$$p_3 = C^n_{n-2}p^{n-2}q^2, \tag{68}$$

and so on, until for exactly r times out of n trials this probability is

$$p_r = C^n_r p^r q^{n-r}. \tag{69}$$

Now each of these separate occurrences is exclusive, for if the event occurs one certain number of times, it cannot occur any other number of times. Hence, by (E) the probability that it will occur n, $n - 1$, $n - 2, \ldots,$ or r times is

$$P' = p_1 + p_2 + p_3 + \ldots + p_r,$$

or, using (66) to (69),

$$P' = p^n + C^n_{n-1}p^{n-1}q + C^n_{n-2}p^{n-2}q^2 + \ldots + C^n_r p^r q^{n-r}. \tag{70}$$

Now let us consider any one of these coefficients C^n_{n-1}, C^n_{n-2}, C^n_{n-r}, etc. By examining any binomial expansion with positive integral index and comparing the coefficients as counted from either end, for example $(a + b)^6$ in article 53, it will be seen that they occur in the same order in each direction; that is, $C^n_{n-1} = C^n_1$; $C^n_{n-2} = C^n_2$; $C^n_{n-r} = C^n_r$; etc. Therefore, (70) is the same as

$$P' = p^n + C^n_1 p^{n-1}q + C^n_2 p^{n-2}q^2 + \ldots + C^n_{n-r}p^r q^{n-r}. \tag{71}$$

Comparing this expression for P' with (58), it is seen at once that the sum on the right is the same as the sum of the terms up to the $[n - (r + 1)]$th in (58). Therefore,

(G) *The probability that such an event as that of* (F) *will occur at least r times in n trials is the sum of the first $n - r + 1$ terms of the binomial expansion of $(p + q)^n$, namely,*

$$p^n + C^n_1 p^{n-1}q + C^n_2 p^{n-2}q^2 + \ldots + C^n_{n-r}p^r q^{n-r},$$

where $q = 1 - p$.

As an example, what is the chance that the ace will turn up not less than twice in five throws of one die?

Here $n = 5$, $r = 2$, $p = \frac{1}{6}$, $q = 1 - \frac{1}{6} = \frac{5}{6}$. Hence, according to formula (71), using $n - r + 1 = 4$ terms of the binomial expansion,

$$\begin{aligned} P &= (\tfrac{1}{6})^5 + 5(\tfrac{1}{6})^4\cdot(\tfrac{5}{6}) + 10(\tfrac{1}{6})^3\cdot(\tfrac{5}{6})^2 + 10(\tfrac{1}{6})^2\cdot(\tfrac{5}{6})^3 \\ &= \tfrac{1}{7776} + 5\cdot\tfrac{1}{1296}\cdot\tfrac{5}{6} + 10\cdot\tfrac{1}{216}\cdot\tfrac{25}{36} + 10\cdot\tfrac{1}{36}\cdot\tfrac{125}{216} \\ &= \tfrac{1}{7776} + \tfrac{25}{7776} + \tfrac{250}{7776} + \tfrac{1250}{7776} = \tfrac{1526}{7776} \\ &= \tfrac{763}{3888}. \end{aligned}$$

$$\therefore \quad P = \tfrac{1}{5} \text{ very nearly.}$$

That is, the chance of *at least* two aces in five throws is 1 in 5 and not 2 in 5 as might be thought, the odds being 5 to 1 against it.

Chapter 23

BOOLEAN AND OTHER ALGEBRAS

133. Introduction. The algebra discussed up to this point in this book can be shown to follow from the application to numbers (as represented by symbols such as a, b, c . . . , . . . x, y, z or other letters) of operations based on simple axioms and theorems derived from them. These axioms have been very simply stated by John L. Kelly in the following form.*

	Axioms for Addition	Axioms for Multiplication
Closure Axiom	For each number x and each number y, $x + y$ is a number.	For each number x and each number y, $x \cdot y$ is a number.
Commutative Axiom	For each number x and each number y, $x + y = y + x$.	For each number x and each number y, $x \cdot y = y \cdot x$.
Associative Axiom	For all numbers x, y, and z, $(x + y) + z = x + (y + z)$	For all numbers x, y, and z, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Identity Axiom	There is a number a such that $a + y = y$ for every number y.	There is a number c such that $c \cdot y = y$ for every number y.
	{It is important to note that theorems derived from this axiom identify a with 0 and c with 1.}	
Axiom of Inverses	For every number m other than 0 there is a number n such that $m + n = 0$	For every number m there is a number n such that $m \cdot n = 0$
Distributive Axiom	For all numbers x, y, and z, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$	

* John L. Kelly, "Introduction to Modern Algebra," Van Nostrand, 1960.

An important point is that subtraction, division and other algebraic operations follow from these axioms, by means of theorems that are readily derived. Of course, in extending this algebra to real and other numbers, certain other axioms are necessary, e.g. the axiom of order. These further axioms, are not, however, discussed here, in order to proceed at once to the other algebras.

134. The Algebra of Boole. The first statement of Boole's Algebra appeared in the pamphlet "The Mathematical Analysis of Logic" by George Boole, published in 1847. This algebra differed from that already discussed in certain fundamental respects. (1) The symbols denoted classes rather than numbers, since it was designed for the mathematization of logic. (2) The axioms of ordinary, "numerical" algebra were retained essentially in the form stated above. (3) Another axiom was added, called idempotency, expressed by the property that $x \cdot x = x^2 = x$, which marks the basic difference between ordinary algebra, and the algebra of Boole. That algebra, when applied to numbers, is restricted by this property to 0 and 1.

This restriction does not divest a mathematical system of practical applications, as is evident from the fact that binary arithmetic* employs only 0 and 1, and is generally used in digital computers. Moreover, these two symbols, defined somewhat differently from their mathematical use, are the basis of the switching algebra variety of Boolean algebra to be discussed later in this chapter.

As stated in (1) above the letter symbols in Boole's algebra, e.g., x, y and z, denote classes of things—which is consistent with the fact that much, if not all, of classical logic was so formulated. One of its most widely used syllogisms was of the form; dogs are mammals; Airedales are dogs; therefore Airedales are mammals. Clearly this series of statements deals with classes, which may be symbolized by x's, y's and z's as well as by their names, and which can then be expressed in general formulations that apply to statements about plants or minerals or chemicals as well as about dogs. Furthermore these symbols can then be manipulated by methods analogous to those of ordinary algebra.

Thus the operation in Boole's system analogous to multiplication in ordinary algebra was election. By using the symbol 1 to mean the universal class in which all other classes are contained, and the sign

* See Thompson, "Arithmetic for the Practical Man," 3rd Edition, Van Nostrand, 1962.

to mean election, then $1 \cdot x = x$ means election, from the universal class, 1, of the class x. Similarly, $x \cdot y$ or xy means election of the class x from the universal class, followed by election from the class x of those members of class x that are also members of class y.

Extending this concept of election to $x \cdot x$, the interpretation of this expression would be selection of the members of class x which are also members of the class x. Since the class x is unchanged by this second selection of it, the result is written $x \cdot x = x$, the property of idempotency mentioned in (3) above, in which the algebra of Boole differs so markedly from ordinary algebra.

The operation in Boole's system that is analogous to addition in ordinary algebra is "either . . . or." Thus $x + y$ means the class composed of members of class x and members of class y, but not members of both classes.

The operation that is analogous to subtraction is exception. Thus $x - y$ means the class of members of class x except those that are also members of class y.

The symbol 0 is used in Boole's system for the empty class, that is, nothing. This introduces the algebraic relationship $0 \cdot x = 0$, expressed as "electing the members of class x from the empty class yields no members at all."

135. The Algebra of Logic. Throughout the preceding discussion the topic was referred to as Boole's system, rather than Boolean algebra, for changes have been introduced into it by succeeding generations of mathematical logicians. These changes have extended, not only to the notation, but to various of the basic concepts. Thus, in current Boolean systems, $x + y$ is taken to include the class of members of both class x and class y as well as to members of classes x and y only. Moreover, Boole's use of symbols from arithmetic or ordinary algebra has largely been superseded by specific, and more exactly defined symbols, such as those used below:

Λ, which means the empty set, replacing Boole's zero.

V, which means the universal set, the set of all things.

$A \cup B$, which means the set of all things which belong at least to A or to B, called the union of A and B, replacing Boole's $A + B$, when the members common to both are included. If the common members are excluded, symbol $\vee$ is used (disjunction).

$A \cap B$, which means the set of all things which belong both to A and to B, called the intersection of A and B and replacing Boole's AB.

$=$, which means equivalence in the sense of identity.

$\neq$, which means non-equivalence or non-identity.

$\sim B$, which means the set of things that do not belong to B (complement of B).

$A \sim B$, which means the set of things which belong to A but not to B, called the difference of A and B, replacing Boole's $A - B$.

$A \rightarrow B$, which means that A implies B, that is, if A is, true then B is true, often used in the sense that A is true only if B is true.

Using these nine symbols, E. V. Huntington* derived the system of thirty-one principles given below.

LIST OF SET IDENTITIES AND OTHER PRINCIPLES

(1) $A \cup \Lambda = A$

(2) $A \cap \mathrm{V} = A$

(3) $A \cup B = B \cup A$

(4) $A \cap B = B \cap A$

(5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(6) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(7) $A \cup \sim A = \mathrm{V}$

(8) $A \cap \sim A = \Lambda$

(9) $A \cup A = A$

(10) $A \cap A = A$

(11) $A \cup \mathrm{V} = \mathrm{V}$

(12) $A \cap \Lambda = \Lambda$

(13) $\Lambda \neq \mathrm{V}$

(14) $\sim\sim A = A$

(15) $A = \sim B \rightarrow B = \sim A$

(16) $A \cup B \neq \Lambda \rightarrow A \neq \Lambda \vee B \neq \Lambda$

(17) $A \cap B \neq \Lambda \rightarrow A \neq \Lambda$

(18) $A \cup (B \cup C) = (A \cup B) \cup C$

(19) $A \cap (B \cap C) = (A \cap B) \cap C$

(20) $A \cup (A \cap B) = A$

(21) $A \cap (A \cup B) = A$

(22) $\sim A \neq A$

(23) $\sim(A \cup B) = \sim A \cap \sim B$

(24) $\sim(A \cap B) = \sim A \cup \sim B$

(25) $A \sim A = \Lambda$

(26) $A \sim (A \cap B) = A \sim B$

(27) $A \cap (A \sim B) = A \sim B$

(28) $(A \sim B) \sim B = A \sim B$

(29) $(A \sim B) \sim A = \Lambda$

(30) $(A \sim B) \cup B = A \cup B$

(31) $(A \cup B) \sim B = A \sim B$

The first eight of the above principles can be used (with the addition of the axiom "there is an A and there is a B such that A is not equiva-

* E. V. Huntington, Trans. Amer. Math. Soc., 5, 288–309, 1904.

lent to B") to derive the remaining twenty-three principles as theorems. Moreover, any model of these axioms is called a Boolean algebra.* As a means of familiarizing himself with the logical application of Boolean algebra, the reader is urged to write out the meanings of these thirty-one principles in words and to check his results with the following list.

(1) The union of A and the empty set equals A.
(2) The intersection of A and the universal set equals A.
(3) The union of A and B equals the union of B and A.
(4) The intersection of A and B equals the intersection of B and A.
(5) The union of A and (the intersection of B and C) equals the intersection of (the union of A and B) and (the union of A and C).
(6) The intersection of A and (the union of B and C) equals the union of (the intersection of A and B) and (the intersection of A and C).
(7) The union of A and not A equals the universal set.
(8) The intersection of A and not A equals the empty set.
(9) The union of A and itself equals A.
(10) The intersection of A and itself equals A.
(11) The union of A and the universal set is the universal set.
(12) The intersection of A and the empty set equals the empty set.
(13) The empty set is not equal to the universal set.
(14) The complement of the complement of A equals A.
(15) The statement that the set of things belonging to A does not belong to B implies that the set of things belonging to B does not belong to A.
(16) The statement that the union of A and B is not equal to the empty set implies the disjunction of the statements that A is not equal to the empty set and B is not equal to the empty set.
(17) The statement that the intersection of A and B is not equal to the empty set implies that A is not equal to the empty set.
(18) The union of A and (the union of B and C) equals the union of (the union of A and B) and C.
(19) The intersection of A and (the intersection of B and C) equals the intersection of (the intersection of A and B) and C.
(20) The union of A and (the intersection of A and B) equals A.
(21) The intersection of A and (the union of A and B) equals A.
(22) The complement of A is not equal to A.

* Patrick Suppes, "Introduction to Logic," Van Nostrand, 1957.

(23) The complement of the union of A and B equals the intersection of the complement of A and the complement of B.
(24) The complement of the intersection of A and B equals the union of the complement of A and the complement of B.
(25) The difference of A and itself equals the empty set.
(26) The difference of A and (the intersection of A and B) equals the difference of A and B.
(27) The intersection of A and (the difference of A and B) equals the difference of A and B.
(28) The difference of (the difference of A and B) and B equals the difference of A and B.
(29) The difference of (the difference of A and B) and A equals the empty set.
(30) The union of (the difference of A and B) and B equals the union of A and B.
(31) The difference of (the union of A and B) and B equals the difference of A and B.

Many of the above principles will be found to express fundamentals which are widely known under other names. Thus the commutative axioms of addition and multiplication, stated at the beginning of this chapter, are expressed in (3) and (4) above; the distributive axiom, in (5) and (6); the associative axioms, in (18) and (19); the axioms of inverses, in (7) and (8); while (14) is also called the law of involution, and (23) and (24), the laws of dualization or de Morgan's laws. Finally, (10) and (9) are the idempotent laws, already characterized as fundamental to a Boolean algebra.

136. The Algebra of Switching. In applying Boolean algebra to switching problems, it is found convenient in writing equations to use a different notation from that used for logic, as given in the preceding paragraphs. The principles, however, are unchanged, and the thirty-one forms of them stated in the preceding pages, apply with equal validity to circuit problems.

The chief difference in notation is (1) the use of the bar over a symbol or expression to mean its negative or complement, thus $\overline{A}$ means the complement of A, (instead of the expression $\sim A$); (2) the use of the symbol "$\cdot$" or simply the indication of multiplication by writing two terms together, as $A \cdot B$ or AB, to indicate the concept of intersection (instead of the symbol $\cap$); and (3) the use of the symbol $+$ or $\vee$ to indicate the concept of union (instead of the symbol $\cup$).

Since in a Boolean algebra the only values A and B can take is 1 or 0, then the table of binary combination is as follows:

A	B	$A \vee B$	AB
0	0	0	0
1	0	1	0
0	1	1	0
1	1	1	1

That is, $A \vee B = 0$ means that both A and B are 0, $A \vee B = 1$ means that either A or B or both are 1. On the other hand $AB = 1$ means that both A and B are 1.

The direct applicability of this algebra to switching circuits is shown

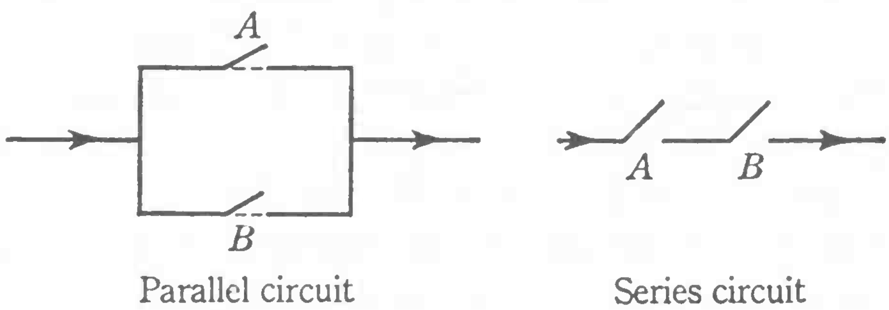


FIG. 2.

by a simple example. Figure 2 shows two simple switching devices, mechanical or electronic, connected in parallel and in series. Consider that they are used to control a current that is to operate another device when passed. Denote the condition of switches A and B by 0 if they are open and by 1 if they are closed. Then the parallel connection will pass current if either A or B (or both) is closed, while the series connection will pass current only if both switches are closed. Thus there is perfect correspondence between the parallel connection and the $A \vee B$ relationship of Boolean algebra, and between the series connection and the AB of Boolean algebra, as can be seen from the table.

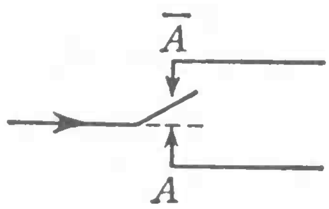


FIG. 3.

Similarly the complement or negative of A, the "not A," is written $\bar{A}$, has the corresponding circuit shown in Figure 3 which shows a two-position switch that actuates one circuit when A is in one position and another when A is in the other position.

It follows that the principles of Boolean algebra explained earlier in this chapter apply to circuit calculations. To facilitate their use for this purpose, the most important of them are restated below in the notation of switching algebra, with reference to the corresponding principle of logical algebra.

$$A \vee 0 = A, \quad \text{Principle (1)}$$
$$A \vee 1 = 1, \quad \text{Principle (11)}$$

Note that 1 is the symbol for the universal set, and 0 for the empty set.

$$A \vee A = A, \quad \text{Principle (9)}$$
$$A \cdot 0 \quad \text{or} \quad A0 = 0, \quad \text{Principle (12)}$$
$$A \cdot 1 \quad \text{or} \quad A1 = A, \quad \text{Principle (2)}$$
$$A \cdot A \quad \text{or} \quad A^2 = A, \quad \text{Principle (10)}$$

Note also that both operations are commutative, that is, $A \vee B = B \vee A$ and $AB = BA$, by Principles (3) and (4) respectively. Both operations are associative, that is, $A \vee (B \vee C) = (A \vee B) \vee C$ and $A(BC) = (AB)C$, by Principles (18) and (19) respectively. Each operation is distributive in regard to the other, that is, $A \vee (BC) = (A \vee B)(A \vee C)$ and $A(B \vee C) = AB \vee AC$ by Principles (5) and (6).

By applying these relationships and operating as in ordinary algebra, switching equations can be modified and simplified, as in ordinary algebra. For example, simplify the expression $A(A \vee B)$.

$$A(A \vee B) = AA \vee AB,$$

by distributive rule (Principle 5)

$$AA \vee AB = A \vee AB,$$

by Principle (9).

$$A \vee AB = A(1 \vee AB),$$

by Principle (2).

$$A(1 \vee AB) = A(1 \vee B),$$

by Principle (6).

$$A(1 \vee B) = A \cdot 1,$$

by Principle (11).

$$A \cdot 1 = A,$$

by Principle (2).

Summarizing, $A(A \vee B) = AA \vee AB = A \vee AB =$

$$A(1 \vee AB) = A(1 \vee B) = A \cdot 1 = A.$$

The application of the complement or "not" function also makes use of these principles.

Thus, $A \vee \bar{A} = 1$ (Principle 7); $A\bar{A} = 0$ (Principle 8); and $A = \bar{\bar{A}}$ (Principle 14).

Another group complement relationship is

$$\overline{A + B + C} = \bar{A}\bar{B}\bar{C}.$$

To illustrate the translation of a switching circuit into its corresponding algebraic function, consider the circuit of Figure 4.

In accordance with the rule already stated, the four switches in the upper branch are represented by the term $ABCD$, while the two switches in parallel, E and F, are represented by the term $(E \vee F)$. Since they are in series with the switch G, the term for the entire lower branch is $(E \vee F)G$, and since the upper branch and the lower branches

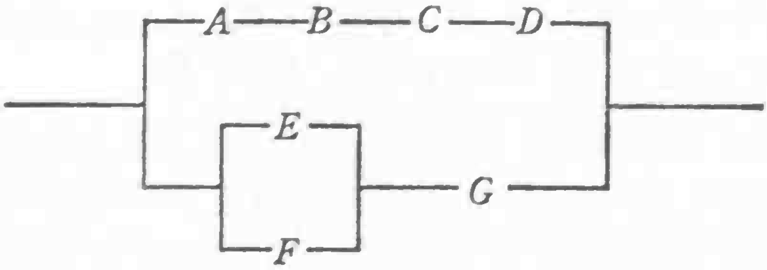


FIG. 4.

are in parallel, the function for the entire circuit is $ABCD \vee (E \vee F)G$.

The electrical devices that are equivalent to Boolean operations are not restricted, of course, to the simple switches shown in the illustrations used here. As stated earlier, they may be vacuum tubes or transistors, and they may be connected in other ways than simple series or parallel circuits. Relays, bridge circuits and many other arrangements are widely used. Whatever their nature or arrangement, however, they perform the basic operations of "or" $(A \vee B)$, "and" (AB) and "not" ($\bar{A}$, $\overline{AB}$, $\overline{A + B}$ etc.), and various combinations of these used for various purposes in switching systems, as for example in the computation or logical operations of digital computers.

As an example of a "not" element computation, consider the circuit shown in Fig. 5a. Here the A-not A switch is in parallel with B, giving the Boolean representation $A(\bar{A} + B)$, which by simple clearing of parentheses gives $A\bar{A} + AB$, and since by Principle (25) $A\bar{A} = \Lambda$ (or 0), then $A\bar{A} + AB = AB$ (Fig. 5b). This simplification illustrates the procedure for treating negatives in circuit algebra.

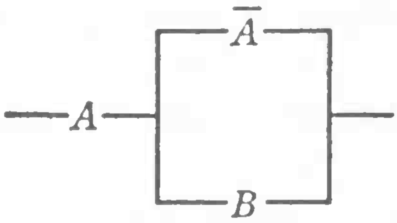


FIG. 5A.

FIG. 5B.

One of the most important applications of switching algebra is to simplify the design of a switching circuit to eliminate unnecessary components. As an example, take the switching circuit that is represented by the Boolean function $ABC \vee A\bar{B}C \vee A\overline{BC}$.

Factoring, $A(BC \vee \bar{B}C \vee \overline{BC})$.
Factoring further, $A[C(B \vee \bar{B}) \vee \overline{BC}]$.
By Principle (7) $(B \vee \bar{B}) = 1$, giving $A(C \vee \overline{BC})$.
By Principle (5) $(C \vee \overline{BC}) = (C \vee \bar{C})(C \vee \bar{B})$, giving $A[(C \vee \bar{C})(C \vee \bar{B})]$.
By Principle (7), $(C + \bar{C}) = 1$, giving $A(C \vee \bar{B})$.

Note that the simplified Boolean function has only three symbols instead of nine in the original, indicating a corresponding reduction in the number of components required in the equivalent circuit.

It is to be noted that simple algebraic factoring can often effect reduction of the number of equivalent components. As an example, consider the function $AC \vee AD \vee BC \vee BD$ which factors into $A(C \vee D) \vee B(C \vee D)$ and thus into $(A \vee B)(C \vee D)$, so reducing the number of components from eight to four.

137. Vectors. Many physical quantities, such as velocities and forces, are directed. To follow the course of a moving object, for example, it is necessary to know not only its speed, in miles per hour, feet per second, or other units, but the direction in which it is moving. Thus, the statement that a steamer is moving toward the northeast at 30 miles per hour is a complete description of its velocity.

While these directed quantities cannot be expressed by single numbers alone, they can be expressed by lines or by pairs of numbers. Such lines are called vectors, and Figure 6 illustrates a vector. It represents a velocity of 30 miles per hour in the northeast direction, because its length is 6 units to a scale of 1 unit = 5 miles per hour and because its line of action has been established by measuring an angle that is half of the angle NOE between north and east. Since angle NOE, being a right angle, is 90°, then the angle between north

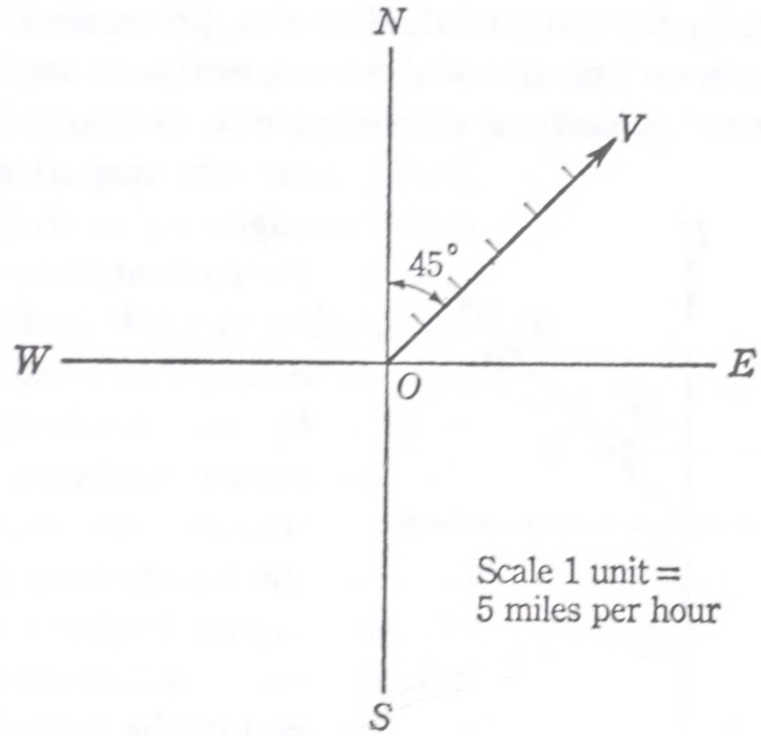


FIG. 6.

and northeast is 45°; and the vector OV being directed to the northeast, makes an angle of 45° with the axis ON.

Suppose now that the engine speed of the steamer and its heading remain unchanged, but that it is moving in a current, such as the Gulf Stream, which at that point has a northeast velocity of 10 miles per hour. Figure 7 shows the vector of the current velocity, $O'C$, which

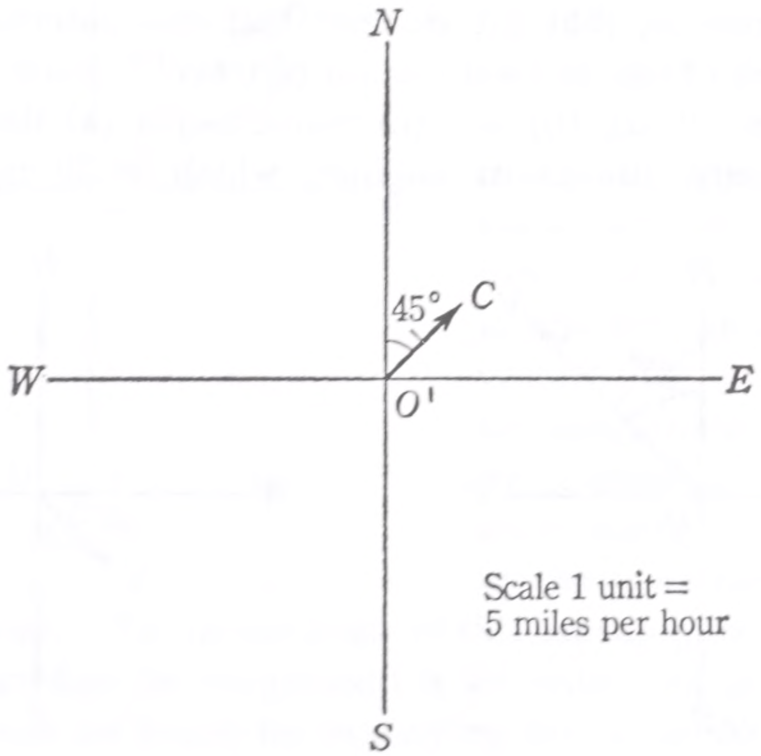


FIG. 7.

has been drawn by the same method as was the vector OV in Figure 6, which was the vector velocity of the steamer due to its engines.

Now the overall velocity of the steamer is obviously the sum of the velocity imparted to it by its engines and that imparted by the current, that is, the sum of vectors OV in Fig. 6 and $O'C$ in Fig. 7. They may be added by the geometric method of vector addition, which is to transfer the second vector to the diagram of the first, placing the beginning of the second vector at the same point as the end of the first vector, keeping their lengths and directions unchanged, as has been done in Fig. 8. Then the line from the beginning of the first vector to the end of the second, that is, the line OC, is the vector sum of the two vectors OV and $O'C$. Since OV is 30 miles per hour, and $O'C$ is 10 miles per hour, both in the same direction, obviously OC is 40 miles per hour in that direction, that is, northeast.

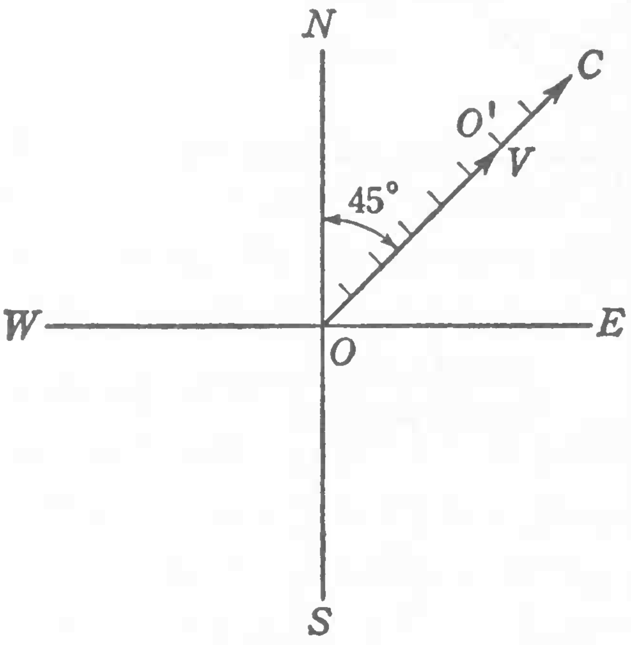


FIG. 8.

Suppose, however, that the steamer had encountered an opposing current, instead of one in its direction of travel. Such a case is illustrated in Figure 9, (a), (b) and (c) which shows (a) the vector of the steamer's velocity due to its engines, which is 30 miles per hour

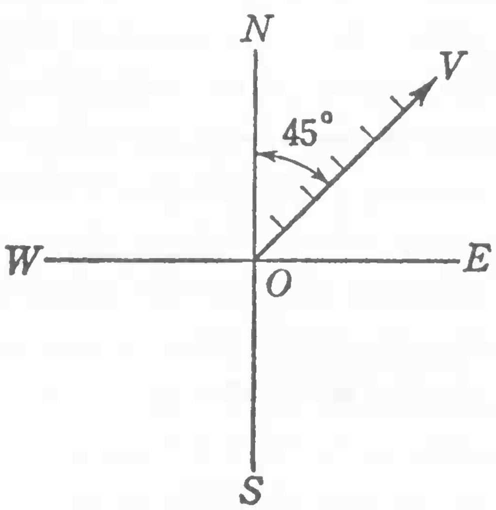


FIG. 9A.

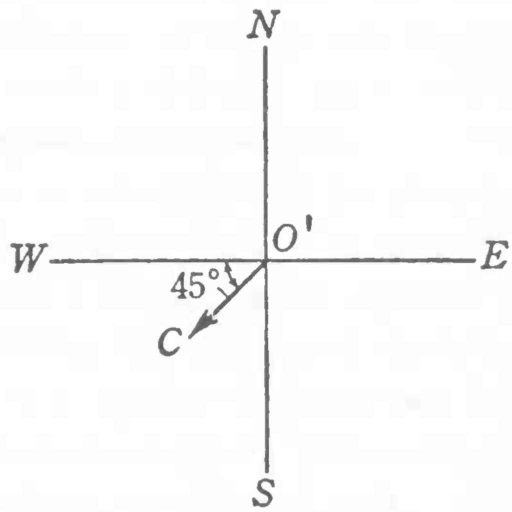


FIG. 9B.

northeast, as before; (b) the vector of current velocity, $O'C$ which is now 10 miles per hour but southwest instead of northeast, and (c) the vector sum OC, obtained as before by placing the vector $O'C$ so that its beginning O' is at the same point, V, as the end of vector OV, and with the length and direction of the vectors unchanged. The line from the beginning of the first vector to the end of the second, OC is now $30 - 10 = 20$ miles per hour in the northwest direction.

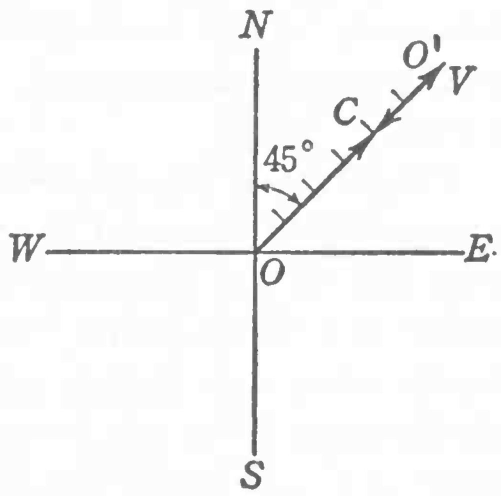


FIG. 9C.

Finally, suppose that the velocity of the current is not directed along the same line as that of the engines, but in another direction, e.g., toward the south, but still at 10 miles per hour. Then the method of vector addition, that is, placing the beginning of the second vector at the end of the first, brings the end of the second vector off the line of the first, as shown in Fig. 10. However, the rule of vector addition, that the vector sum is a line from the beginning of the first to the end of the second vector, still applies. This situation is illustrated in Fig. 10, which shows that the vector sum of an engine velocity of 30 miles per hour northeast, represented by the vector OV, and a current velocity of 10 miles per hour southward, represented by a vector $O'C$, is a vector OC. Its magnitude can be found by measuring its length in scale units, and multiplying by the scale ratio of 1 unit = 5 miles per hour. To the accuracy of this drawing, its length is about 4.8 units; therefore its magnitude is 24 miles per hour. Similarly its direction can be found by measuring the angle NOC with a protractor, which is found to be about 62°, that is, its direction is

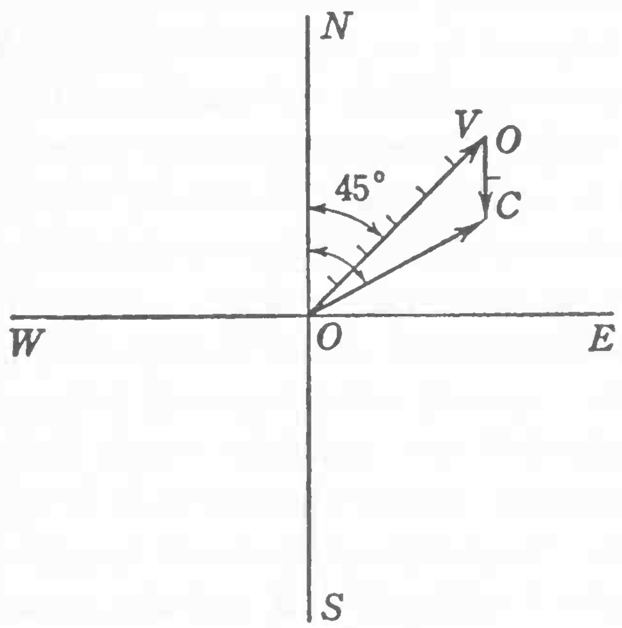


FIG. 10.

62° west of north. This vector is the actual velocity of the steamer.

This operation of adding vectors graphically can be performed to any desired degree of accuracy by using trigonometric methods and tables. However, a similar result can be obtained by algebraic processes, which have as their basis the fact that a two-dimensional vector can be represented, not only by a line, but also by a pair of numbers.

138. Vector Algebra—Addition. The algebraic system of operating with vectors can be explained by starting with the observation that the vectors discussed in the preceding pages can be described in terms of a single point, the terminal point of the vector, since their initial point was always taken at the origin O. Thus, by describing the position V, we can determine the vector OV, and by describing the position C, the vector OC, and so on.

Since these are two-dimensional vectors (because they do not move off the paper into space), their position could be described in terms of numbers measured along two axes. Thus the V, the terminal point of the vector OV, could be established by measuring the distance along the OE axis to a point vertically below V, and then by measuring the distance along the ON axis to a point horizontally in line with V, as shown in Fig. 11. The two measurements, which might be called v-east and v-north, or v_E and v_N, are found to be about 4.2 units each. From them, we could construct the vector OV, or we could construct any two-dimensional vector from similar measurements, by reversing the operations. That is, we would lay off a distance v_E along the OE axis, and v_N along the N axis, erect perpendicular lines to both axes at those points, and find the point V at their intersection.

Thus in vector algebra, any two-dimensional vector is described by two numbers. They could be written merely with a comma between them, e.g., by the numbers 4.2, 4.2 for OV, or 4.2, 2.2 for vector OC of Fig. 10, by remembering the first number always means the E-W axis and the second number, the N-S axis. In the usual algebraic notation, the first number would be measured along the X-axis and the second number along the Y-axis.

Then by the use of this notation of a pair of numbers to represent a vector, vector OV (Fig. 11) is written 4.2, 4.2; and vector OC (Fig. 10) is written 4.2, 2.2. To generalize, any two-dimensional vector is written a, b; where a is the projection (component) of the vector on the X-axis, and b, its projection (component) on the Y-axis.

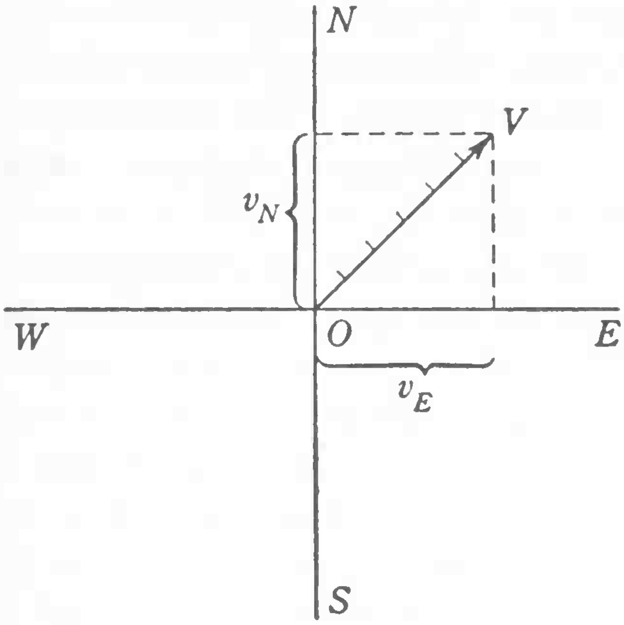


FIG. 11.

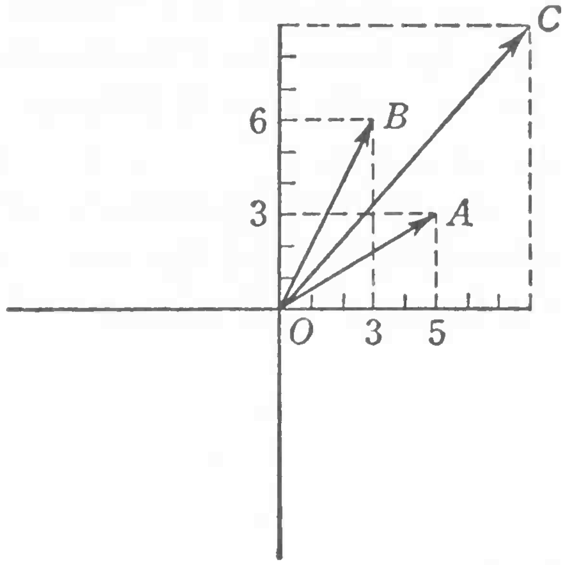


FIG. 12.

With vectors expressed in this way, they can be added by direct algebraic methods. The rule is as follows: The sum of the vectors a, b and c, d is the vector $a + b$, $c + d$. For example, the vector OA of Figure 12 is represented by the coordinates 5, 3, and the vector OB is represented by the coordinates 3, 6, therefore the sum of OA and OB is a vector OC having the coordinates $5 + 3$, $3 + 6$ or 8, 9.

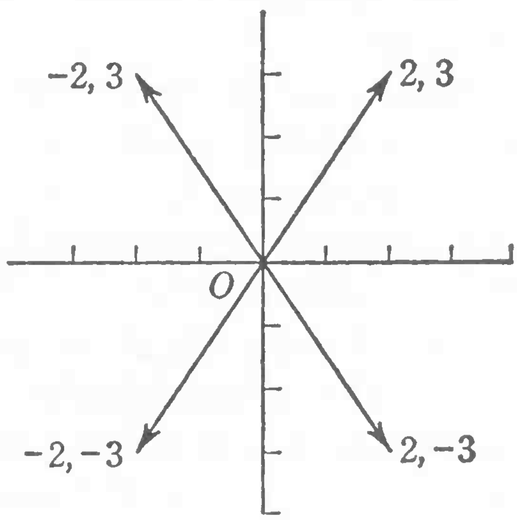


FIG. 13.

In applying the algebraic notation to vectors, components along the X-axis measured to the right of the origin are represented by positive numbers, while components measured to the left are represented by negative ones. Similarly components along the Y-axis are represented by positive numbers if they are above the origin, and by negative numbers if they are below it. Figure 13 represents the vectors 2, 3; 2, -3; -2, -3; and -2, 3, thus showing the positions of two-dimensional vectors with the four possible combinations of positive and negative components.

An important advantage of the algebraic method of vector addition over the graphical one is the obviously greater convenience and accuracy of adding numbers algebraically rather than adding lines by

constructing drawings. This advantage is especially great when several vectors are to be added. The graphical method would require the correct graphical representation of each vector. The simplicity of the algebraic method is illustrated by the following example. Add the vectors 3, 11; −2, 6; 5, −3, 2, −4, and −1, −2.

Simply write the vectors in double column so that the X-components are in one column and the Y-components are in the other, and add the columns algebraically.

$$\begin{array}{rr} 3 & 11 \\ -2 & 6 \\ 5 & -3 \\ 2 & -4 \\ -1 & -2 \\ \hline 7 & 8 \end{array}$$

Thus the vector 7, 8 is the sum, or resultant of the five vectors given.

The magnitude of a vector can be readily found by the sum of squares relationship. Thus the magnitude (length) of the vector OV, whose algebraic representation is a, b, is $\sqrt{a^2 + b^2}$.

For example, the magnitude of the vector 7, 8, which was obtained in the addition example above, is $\sqrt{7^2 + 8^2} = \sqrt{49 + 64} = \sqrt{113} = 10.63 \ldots$.

Note that the magnitude (length) of a vector is independent of its quadrant, that is, whether its components are positive or negative. Thus the vectors 7, −8; −7, −8, and −7, 8 would all have the same magnitude as the vector 7, 8 above, since $(-7)^2 = 49 = (7)^2$ and $(-8)^2 = 64 = (8)^2$.

139. Vector Algebra—Multiplication. The multiplication of a vector by a scalar (that is, an ordinary number) is easily understood as an extension of vector addition. Thus if the vector 2, 3 is added to another vector, also 2, 3, then by the rule for addition already stated, the result is the vector 2 + 2, 3 + 3, or 4, 6, which is equivalent to the vector $2(2, 3) = 4, 6$. The rule for multiplication of a vector by a scalar is simply to multiply each component of the vector by the scalar to obtain a new vector, or in symbolic form, $n(a, b) = na, nb$.

The multiplication of a vector by a vector does not, like addition of vectors or multiplication of vectors by scalars, lend itself readily to illustration by examples involving velocities of moving vehicles. It is, however, of fundamental importance in many scientific and engineer-

ing applications. The laws of electrodynamics, for example, are most simply stated in terms of vector multiplication. A simple illustration is the concept of work, which can be expressed by $\mathbf{F} \cdot \mathbf{d}$, where $\mathbf{F}$ is the force vector and $\mathbf{d}$ the displacement vector (See article 140 for discussion of bold-face notation for vectors).

There are two possible products of the multiplication of one vector by another—the inner, or dot product, expressed as $\mathbf{A} \cdot \mathbf{B}$, and the outer or cross product, expressed as $\mathbf{A} \times \mathbf{B}$. The inner product is considered first in the following discussion.

The rule for inner vector multiplication of two-dimensional vectors is: The inner product of a vector a, b and a vector c, d is written $(a, b) \cdot (c, d)$ and is the scalar number $ac + bd$.

In other words, to find the inner product of two two-dimensional vectors multiply algebraically the numbers representing their X-components, multiply algebraically the numbers representing their Y-components, and add the two products. The result is not a vector, but a scalar (ordinary number).

Example. (i) Find the inner product of the vectors 3, 5 and 4, 10. By the rule, inner product $= (3, 5) \cdot (4, 10) = (3 \cdot 4) + (5 \cdot 10) = 62$.

Example. (ii) Find the inner product of the vectors 3, -5 and 4, 10. By the rule, inner product $= (3, -5) \cdot (4, 10) = (3 \cdot 4) + (-5 \cdot 10) = -38$.

Example. (iii) Find the inner product of the vectors -3, -5 and 4, -10. By the rule, inner product $= (-3, -5) \cdot (4, -10) = (-3 \cdot 4) + (-5 \cdot -10) = 38$.

A useful application of the inner product is its use as a test for perpendicularity, because two vectors are perpendicular if their inner product is zero, and conversely.

Thus the inner product of the vectors 6, 3 and 4.5, $-9 = (6, 3) \cdot (4.5, -9) = (6 \cdot 4.5) + (3 \cdot -9) = (27) + (-27) = 0$. Plotting these vectors shows graphically that they are perpendicular.

The outer or cross product of two vectors, unlike the inner or dot product, is another vector. The rule for vector cross-multiplication is: The cross-product of the two two-dimensional vectors a, b and c, d is a vector having the magnitude $ad - bc$ and a direction perpendicular to the plane determined by the two vectors a, b and c, d.

As an example, consider the vectors 4, 5 and 7, 9, which are taken to be lying in the plane of the paper (the XY-plane). Then their outer or cross product $= (4, 5) \times (7, 9) = (4 \cdot 9) - (5 \cdot 7) = 1$, a vec-

tor of length (magnitude) 1 and a direction pointing straight up from the paper (the Z-axis). (It should be noted that the usage followed here, of taking the positive direction of the Z-axis as vertically upward, is not followed universally).

Example, (i). Find the outer product of the vectors 2, 7 and 3, 5. By the rule, outer product $= (2, 7) \times (3, 5) = (2 \cdot 5) - (7 \cdot 3) = 10 - 21 = -11$, a vector of magnitude 11 directed downward perpendicular to the plane of the paper.

Example, (ii). Find the outer product of the vectors -1, 4 and -6, 2. By the rule, outer product $= (-1, 4) \times (-6, 2) = (-1 \cdot 2) - (4 \cdot -6) = (-2) - (-24) = 22$, a vector of magnitude 22 directed upward perpendicular to the plane of the paper.

A convenient way of cross-multiplication is by writing the vectors in determinant form. Thus Example (ii) above would be written

$$\text{Outer product} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -1 & 4 \\ -6 & 2 \end{vmatrix} = (-1 \cdot 2) - (4 \cdot -6) = 22$$

Example, (iii). Find the outer product of the vectors -5, 2 and 3, -8. Using the determinant form

$$\text{Outer product} = \begin{vmatrix} -5 & 2 \\ 3 & -8 \end{vmatrix} = (-5 \cdot -8) - (2 \cdot 3) = 34, \text{ a vector}$$

of magnitude 34 directed upward along the Z-axis.

140. Three-Dimensional Vectors. Up to this point the discussion has dealt exclusively with two-dimensional vectors. Problems frequently arise in science and engineering that involve vectors of three or more dimensions, and it is fortunate that the mathematical methods for treating them are essentially extensions of those already developed for two-dimensional vectors.

In mathematical usage, vectors are commonly represented by bold-face letters, which are equivalent to the ordered pairs we have been using for two-dimensional vectors and to the ordered triples used for three-dimensional vectors, and so on. Thus a given three-dimensional vector can be represented equally well by the letter **A** as by the ordered triple, a, b, c. To familiarize the reader with the bold-face notation, both forms will be used throughout the remainder of the discussion.

The rule for addition of three-dimensional vectors is simply an extension of that given already for two-dimensional vectors. It is: The sum of the vectors $\mathbf{A} = (a, b, c)$ and $\mathbf{X} = (x, y, z)$ is the vector $\mathbf{A} + \mathbf{X} = (a + x, b + y, c + z)$.

Example. Add the three-dimensional vectors 3, -5, 11 and -2, 6, -5. By the rule, the vector which is the sum of these two vectors is the vector $3 + -2$, $-5 + 6$, $11 + -5$ or 1, 1, 6. It may be shown in three-dimensions, or in two-dimensions in perspective (drawing) by marking off 1 unit to the right of the origin on the X-axis, 1 unit above the origin on the Y-axis, and 6 units above the origin on the Z-axis, and projecting from these marks to find the point 1, 1, 6. The vector is then drawn from the origin to this point.

The magnitude of a three-dimensional vector is found by the sum of squares relationship, as was that of the two-dimensional vector. Thus the magnitude of the three-dimensional vector **V**, whose algebraic representation is a, b, c, is $\sqrt{a^2 + b^2 + c^2}$.

Example. Find the magnitude of the vector 5, 4.5, -6. $V = \sqrt{\overline{5}^2 + \overline{4.5}^2 + \overline{-6}^2} = \sqrt{25 + 20.25 + 36} = \sqrt{81.25} = 9.013\ldots$. (Note that the magnitude of the vector **V** is represented by the lightface letter V.)

Three-dimensional vectors are multiplied by scalars (ordinary numbers) in the same way as are two-dimensional vectors, that is, each component is multiplied by the scalar to obtain the new vector. Symbolically, $n\mathbf{V} = n(a, b, c) = na, nb, nc$.

Example. Multiply the vector 3.3, 4, 6.1 by 3. By the rule, the new vector is $(3\cdot 3.3, 3\cdot 4, 3\cdot 6.1)$ or $(9.9, 12, 18.3)$.

Inner multiplication of three-dimensional vectors is strictly analogous to that of two-dimensional vectors: The inner product of a vector $\mathbf{V} = (a, b, c)$ and a vector $\mathbf{W} = (x, y, z)$ is $\mathbf{V}\cdot\mathbf{W} = (a, b, c)\cdot(x, y, z) = (ax + by + cz)$.

Example. Find the inner product of the vectors 3, -2, 5, and .5, 6, -4. By the rule, the inner product $= (3, -2, 5)\cdot(.5, 6, -4) = (3\cdot .5) + (-2\cdot 6) + (5\cdot -4) = -30.5$.

Outer multiplication of three-dimensional vectors, while analogous to that of two-dimensional vectors, involves more operations, and therefore the use of determinant notation is very helpful. In that form the rule is: The outer-product (cross-product) of the vectors $\mathbf{V} = a, b, c$ and $\mathbf{W} = x, y, z$ is the vector having for its defining numbers (coordinates) the determinants

$$\begin{vmatrix} b & c \\ y & z \end{vmatrix}, \quad -\begin{vmatrix} a & c \\ x & z \end{vmatrix}, \quad \begin{vmatrix} a & b \\ x & y \end{vmatrix}.$$

Note that the determinant formed by the Y- and Z-coordinates

(components) of the vectors multiplied gives the X-coordinate (component) of the cross-product vector, the determinant formed by the X- and Z-coordinates of the vectors multiplied gives the negative of the Y-coordinate of the cross-product vector, and the determinant formed by the X- and Y-coordinates of the vectors multiplied gives the Z-coordinate of the cross-product vector. This relationship corresponds to that in cross-multiplication of two-dimensional vectors, wherein the product vector was perpendicular to the plane of the two vectors multiplied.

Example. Find the outer product (cross-product) of the vectors $\mathbf{V} = 4, 3, 6$ and $\mathbf{W} = 2, -5, -7$. By the rule, the cross-product is the vector

$$\begin{vmatrix} 3 & 6 \\ -5 & -7 \end{vmatrix}, \quad -\begin{vmatrix} 4 & 6 \\ 2 & -7 \end{vmatrix}, \quad \begin{vmatrix} 4 & 3 \\ 2 & -5 \end{vmatrix}.$$

The first determinant $= 3\cdot -7 - (6\cdot -5) = 9$. The second determinant $= 4\cdot -7 - (6\cdot 2) = -40$. The third determinant $= 4\cdot -5 - (3\cdot 2) = -26$. Therefore the product vector is $9, -(-40), -26 = 9, 40, -26$. Since these three numbers are the coordinates of the point which, with the origin defines the vector product, its magnitude (length) is found by the sum of squares rule to be $\sqrt{a^2 + b^2 + c^2}$ $= \sqrt{9^2 + \overline{40}^2 + \overline{-26}^2} = \sqrt{81 + 1600 + 676} = \sqrt{2357} = 48.54\ldots$.

A useful concept in vector calculations is that of the unit vector. A convenient notation for applying this concept is that using the symbols **i**, **j**, and **k** for the unit vectors along three mutually perpendicular axes. Where these are taken as the Cartesian X-, Y-, and Z-axes, the unit vector **i** applies to a vector, or a component of a vector, along the X-axis; the unit vector **j** applies to one along the Y-axis, and the unit vector **k**, to one along the Z-axis.

When, for example, 6**i** is a vector of 6 units along the X-axis, 3**j** is a vector of 3 units along the Y-axis, and 4**k** is a vector of 4 units along the Z-axis. The negative sign, as before, indicates reversal of direction of a vector. Thus 6**i** is a vector of six units to the right along the X-axis, while -6**i** is a vector to the left along the X-axis.

Using this notation, the numbers representing a vector are not dependent upon their order, as they were in the case of the notation used up to this point in this chapter. Instead of writing a three-dimensional vector as, for example, 5, 2, 7, one would write it as 5**i**, 2**j**, 7**k**, or in general terms, as a**i**, b**j**, and c**k**, where a**i**, b**j** and c**k** are the

components of the vector along the X-, Y-, and Z-axis, or in other words, where a, b and c are respectively, the X-, Y-, and Z-coordinates of its terminal point (O, the origin, being its initial point).

In performing vector calculations with this notation, it is important to establish the nature and relation of the unit vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. *They are not to be confused with* $\sqrt{-1}$ *for which the symbol* i *or* j *was used earlier in this book.* In fact, the unit vectors cannot be expressed for multiplication as arithmetical numbers, but their products are so defined in accordance with the laws of vector multiplication, as follows:

$$\begin{array}{rcr}
\mathbf{i}\cdot\mathbf{i} = 1 & \mathbf{j}\cdot\mathbf{j} = 1 & \mathbf{k}\cdot\mathbf{k} = 1 \\
\mathbf{i}\cdot\mathbf{j} = 0 & \mathbf{j}\cdot\mathbf{i} = 0 & \mathbf{i}\cdot\mathbf{k} = 0 \\
\mathbf{k}\cdot\mathbf{i} = 0 & \mathbf{j}\cdot\mathbf{k} = 0 & \mathbf{k}\cdot\mathbf{j} = 0 \\
\mathbf{i}\times\mathbf{i} = 0 & \mathbf{j}\times\mathbf{j} = 0 & \mathbf{k}\times\mathbf{k} = 0 \\
\mathbf{i}\times\mathbf{j} = \mathbf{k} & \mathbf{j}\times\mathbf{k} = \mathbf{i} & \mathbf{k}\times\mathbf{i} = \mathbf{j} \\
-\mathbf{j}\times\mathbf{i} = \mathbf{k} & -\mathbf{k}\times\mathbf{j} = \mathbf{i} & -\mathbf{i}\times\mathbf{k} = \mathbf{j}
\end{array}$$

These product relationships may be summarized in the following statements: (1) the dot product of two identical unit vectors is 1; the dot product of two different unit vectors is 0; (2) the cross-product of two identical unit vectors is 0; the cross product of two different unit vectors is the third unit vector, which has a sign (+ or −) following the rule of signs given in the above table.

Any vector may be expressed in terms of unit vectors by writing it as a combination of a scalar (ordinary number) and a unit vector. Thus, $a\mathbf{i}$ is a vector of magnitude a directed along the X-axis (in the positive direction, i.e., to the right, if a is positive, and in the negative direction, i.e., to the left, if a is negative); $b\mathbf{j}$ is the corresponding Y-axis vector; $c\mathbf{k}$ is the corresponding Z-axis vector; and $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a three-dimensional vector having the components $a\mathbf{i}$, $b\mathbf{j}$ and $c\mathbf{k}$, that is, having the terminal point a, b, c if its initial point is the origin.

In terms of this unit vector notation, the rules for combination of vectors can be restated in concise form as follows:

The sum of the vectors $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ is the vector $(a + d)\mathbf{i} + (b + e)\mathbf{j} + (c + f)\mathbf{k}$ where the additions are algebraic.

The magnitude (length) of the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is

$$\sqrt{(a\mathbf{i})^2 + (b\mathbf{j})^2 + (c\mathbf{k})^2} = \sqrt{a^2 + b^2 + c^2},$$

since $\mathbf{i}\cdot\mathbf{i} = 1$, $\mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{k}\cdot\mathbf{k} = 1$.

The scalar product of the vectors $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ is $a\mathbf{i}\cdot d\mathbf{i} + b\mathbf{j}\cdot e\mathbf{j} + c\mathbf{j}\cdot f\mathbf{j}$ = the number $ad + be + cf$, since as shown above the dot products of the identical vectors are unity.

The vector product of the vectors $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ is

$$\begin{vmatrix} b\mathbf{j} & c\mathbf{k} \\ e\mathbf{j} & f\mathbf{k} \end{vmatrix} + \begin{vmatrix} a\mathbf{i} & c\mathbf{k} \\ d\mathbf{i} & f\mathbf{k} \end{vmatrix} + \begin{vmatrix} a\mathbf{i} & b\mathbf{j} \\ d\mathbf{i} & e\mathbf{j} \end{vmatrix}$$

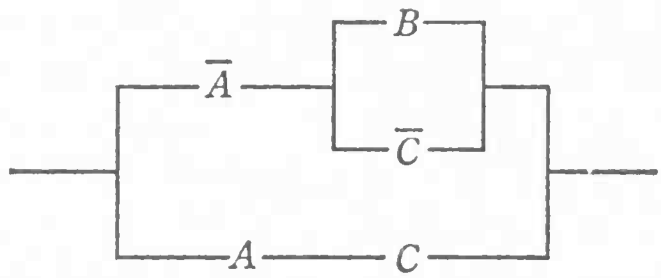
$$= (b\mathbf{j} \times f\mathbf{k} - e\mathbf{j} \times c\mathbf{k}) + (a\mathbf{i} \times f\mathbf{k} - d\mathbf{i} \times c\mathbf{k}) + (a\mathbf{i} \times e\mathbf{j} - d\mathbf{i} \times b\mathbf{j}).$$

Since from the table already given $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $-\mathbf{i} \times \mathbf{k} = \mathbf{j}$ or $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, the expression for the vector product becomes $(bf - ce)\mathbf{i} + (af - cd)-\mathbf{j} + (ae - bd)\mathbf{k} = (bf - ce)\mathbf{i} - (af - cd)\mathbf{j} + (ae - bd)\mathbf{k}$, which is the vector product of the two vectors.

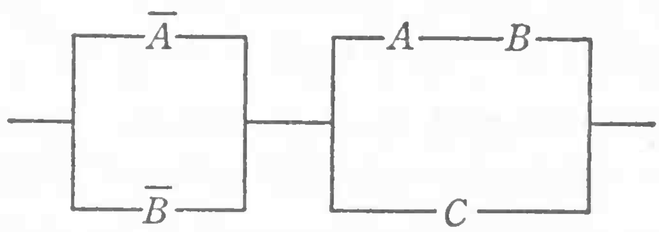
Note that this product is identical, in unit vector notation, with the result obtained in the ordered number notation above.

141. Exercises.

1. Simplify the Boolean equation $(A \vee B)(A \vee C)$.
2. Simplify the Boolean equation $A \vee AB$ and draw the equivalent circuits for both forms.
3. Simplify the Boolean equation $A(A \vee B)$ and draw both equivalent circuits.
4. Simplify the Boolean equation $A \vee \bar{A}B$ and draw both equivalent circuits.
5. Write the Boolean equation for the circuit

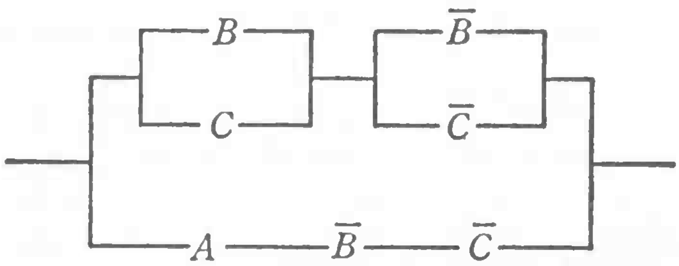


6. For the circuit



write the equation and its simplified form.

7. Write the Boolean equation for the circuit



8. Simplify the equation $\bar{A}\bar{B}C(\bar{D} \vee D) \vee \bar{A}BCD \vee A(\bar{B}C\bar{D} \vee B\bar{C}D)$ and draw the simplified form.

9. Add the vectors 4, 5 and 3, 2.

10. Add the vectors 4, −2 and 5, −1.

11. Find the sum of the vectors 1,2; 6, −4; 8, −1; −3,7; and 2,6, and then find the magnitude (length) of the vector sum.

12. Find the magnitudes of the vectors 3,4; −2,6; −6, −5; and 7, −4.

13. Find the true direction and rate of motion of a ship headed northwest at an engine speed of 5 miles per hour and acted upon by a current moving northeast at 5 miles per hour.

14. Find the inner product of the vectors 4,3 and 7,6.

15. Find the inner product of the vectors −3,6 and 5,2.

16. Find the inner product of the vectors −4, −7 and −3, −9.

17. Find the magnitude and direction of the outer (vector) product of the vectors 4,3 and 8, −7.

18. Find the magnitude and direction of the outer (vector) product of the vectors −5, −6 and 4, −9.

19. Find the sum of the vectors 2,4, −3 and 5,5,5.

20. Find the sum of the vectors 2,8,4; 3, −7, −2; −1,6, −4; −2,4, −3.

21. Find the vector sum and its magnitude for the vectors 3,4,5; −2, −5, −6; and 3,7, −2.

22. Find the dot product of the vectors 2,1,3 and 6,7, −8.

23. Find the dot product of the vectors 4,8,4 and −6, −5, −8.

24. Find the work done by the force expressed by the vector 5,9,2 pounds acting through a displacement expressed by the vector 2,7,3 feet.

25. Find the cross product of the vectors 5,7,9 and −4, −6, −8.

26. Find the cross product of the vectors 6,2,5 and 3,1,7.

27. Find the cross product of the vectors 2,6,5 and 3,4,7.

REVIEW PROBLEMS

1. Evaluate $2x^2 - x^{-1/3} + 3x^0$ when $x = 8$.
2. Simplify: $\dfrac{(x^2y)^4}{x^3} \div 2xy^3$.
3. Find x to the nearest hundredth: $5^x = 24$.
4. Solve for x: $x^{3/2} - 8 = 0$.
5. Solve for x: $5^{x+3} = 25^{2x-1}$.
6. Find the sum of the first 50 terms of the progression 3,7,11, ...
7. a. Find the sixth term of the geometric progression 5,10,20, ...
   b. Find the sum of the first 6 terms of this progression.
8. a. Find the arithmetic mean of 27 and 3.
   b. Find the geometric mean of 27 and 3.
9. Solve and check: $$\begin{aligned} 2x - y &= 4. \\ 4x + 3y &= -7. \end{aligned}$$
10. Solve and check: $$\begin{aligned} x + y + z &= 2. \\ 2x + 3y - z &= 3. \\ 3x + y + 2z &= 6. \end{aligned}$$
11. A motorist starts at 9:00 A.M. from a certain point, traveling at 50 miles per hour. An hour later a second motorist leaves the same point and travels at a rate of 55 miles per hour. At what time will the second motorist be 10 miles behind the first?
12. A gourmet shop wishes to sell 100 pounds of Mocha-Java coffee for \$3.25 per pound. If Mocha coffee sells for \$3.40 per pound and Java for \$3.15 per pound, how many pounds of each should be used in the mixture?
13. A group of skiers wishes to rent a ski house for the winter. They estimate that the cost of the house and the utilities will be \$4,000. If two more people join the group the cost per person will decrease by \$100. How many people were in the original group?
14. A store sells a dress for \$70, making a profit of 40% of the selling price. The buyer wishes to put the dress on sale and make a profit of 30% of the original selling price. What is the sale price of the dress?
15. Given: $f(x) = x^2 - 3x$. If, for example, $f(5) = 5^2 - 3 \cdot 5 = 25 - 15 = 10$, find:
    a. $f(7)$ b. $f(-1)$ c. $f(0)$ d. $f(a)$.
16. Calculate:
    a. ${}_7P_3$ b. ${}_{11}C_2$
17. How many 7-letter words can be formed using all of the letters of the word *Pontoon?*
18. In how many ways can a committee of 4 people be chosen from a group of 20 people?
19. Two fair dice are tossed. Find the probability that the sum of the faces is 7.

20. A fair die is tossed 10 times. Find the probability that there are at most 2 fours showing.
21. In how many different ways can 5 people seat themselves in 12 chairs in a row?
22. A hospital pharmacy has 60 milliliters of a 6% solution of iodine and alcohol. How many milliliters of pure iodine must be added to obtain a 10% solution?
23. Derek went to the bank to change a \$10 bill into a specific number of nickels and quarters. The teller inadvertently interchanged the number of nickels and the number of quarters and Derek found himself 80¢ short. How many nickels and how many quarters did Derek originally request?
24. Bill ran the 26-mile New York Marathon at a rate of 2 miles per hour faster than John. He finished 2 hours 10 minutes earlier. How fast was each runner traveling?
25. Solve for x: $2(3x + 7) + 9 = 14 - 3(4x - 1)$.
26. Solve for x: $\frac{8x - 5}{12} = \frac{x + 6}{3} - \frac{2x - 7}{4}$.
27. Two numbers are in the ratio of 2:5. If 3 is added to the smaller and 5 is subtracted from the larger, the new numbers are in the ratio 1:2. Find the original numbers.
28. If $U = \{\text{integers from 1 to 10}\}$, $A = \{1,2,3,4,5,6\}$, $B = \{3,5,7,9\}$, and $C = \{8,9,10\}$,

    a. List the members of each of the following sets:

    1. $A \cup B$
    2. $A \cap B$
    3. $A \cap C$
    4. $B \cup \overline{C}$
    5. $\overline{A} \cap C$
    6. $B \cap (A \cup C)$

    b. Is $\overline{A \cup C} = \overline{A} \cup \overline{C}$? Justify your answer.
29. Perform the indicated operations and express each answer in simplest form in terms of i:

    a. $\sqrt{-45} + \sqrt{-125}$

    b. $\sqrt{-25}\,\sqrt{-4}$

    c. $\frac{2}{3i}$

    d. $\frac{5 - 2i}{4 + i}$

    e. $(2 + 3i\sqrt{7})^2$
30. Solve for x and y: $(2 - 3i) + (x + yi) = 5 + i$.
31. Which of the following axioms does the equation $a(x + y) = a(y + x)$ illustrate?

    a. Associative axiom of addition

    b. Commutative axiom of multiplication

    c. Commutative axiom of addition

    d. Distributive axiom of multiplication over addition
32. Which of the following operations is not commutative?

    a. Addition b. Subtraction c. Multiplication
33. Rationalize each denominator and simplify:

    a. $\frac{5}{\sqrt{12}}$

    b. $\frac{3 + \sqrt{2}}{5 - \sqrt{2}}$
34. Write the first 4 terms of the expansion of $(a + 3b)^{10}$.
35. Write the entire expansion of $(2x - y)^5$.
36. Write the fifth term of the expansion of $(2x + \frac{1}{y})^{12}$.

37. Perform the indicated operations and express the result in simplest form:

$$\frac{x^3 - x^2}{4x^2 - 25} \times \frac{2x^2 + 3x - 5}{9x^3} \div \frac{x^2 - 2x + 1}{6x - 15}$$

38. Factor completely:

a. $6x^5y^2 + 4xy^3$
b. $x^2 - x - 20$
c. $x^4 - 1$
d. $5x^2 - 13x + 6$
e. $x^3 + 8$
f. $2r + 2t - xt - xr$

39. Express each radical in simplest form:

a. $\sqrt[3]{54x^3y^4}$
b. $\sqrt[4]{16a^6b^7}$

40. Perform each operation and simplify:

a. $5\sqrt{108} - \sqrt{300} + 2\sqrt{\frac{1}{3}}$
b. $(5 + \sqrt{3})^2$
c. $(5\sqrt{3})^2$
d. $(5 + \sqrt{3})(5 - \sqrt{3})$

41. The formula for the motion of a pendulum is $t = \pi\sqrt{\frac{l}{g}}$ where t = the number of seconds, l = the length of the pendulum, g = 32.2 feet per second^2 and $\pi = 3.14$.

a. Express log t in terms of log l and log g.
b. If l = 15 cm., find t to the nearest tenth of a second. Use logs to calculate the answer and then check your answer using a calculator.

42. If $\log x = a$ and $\log y = b$, express in terms of a and b:

a. $\log xy$
b. $\log 100x$
c. $\log \frac{1}{10}x$
d. $\log x^2y$
e. $\log x\sqrt{y}$
f. $\log \sqrt[3]{xy}$
g. $\log \frac{x}{y}$

43. Solve for x and check: $\dfrac{10x}{x^2 - 4} - \dfrac{5}{x - 2} = 1.$

44. A plane traveling a distance of 2,300 miles with the wind takes 4 hours 36 minutes to make a certain trip. The return trip takes 5 hours. Find the rate at which the wind is blowing.

45. Solve by factoring: $6x^3 - 11x^2 + 4x = 0$.

46. Find to the nearest hundredth the roots of $3x^2 + 4x = 1$.

47. Solve and express each root in the form $a + bi$:

$$x^2 - 2x + 2 = 0.$$

48. A chemist has 50 cc. of a solution which is 12% salt. How much water must she evaporate to produce a solution which is 15% salt?

49. An investor wishes to decide between two possible ways of investing a sum of \$10,000 for one year. One bank offers an interest rate of 12.5% simple interest while another offers interest of 12% compounded quarterly. Which investment will yield greater earnings? How much more?

50. An apprentice takes two hours more than an experienced mechanic to service an airplane. They work together for 3 hours and then the mechanic is called away and the apprentice completes the job in 1 hour more. How long would it take each man working alone to service the plane?

ANSWERS TO PROBLEMS

1. $130\frac{1}{2}$.
2. $\frac{x^7y}{2}$.
3. 1.97.
4. 4.
5. $\frac{5}{3}$.
6. 5,050.
7. a. 160.
   b. 315.
8. a. 15.
   b. 9.
9. $x = \frac{1}{2}, y = -3$.
10. $x = 1, y = -1, z = 2$.
11. 6 P.M.
12. 40 lbs. of Mocha,
    60 lbs. of Java.
13. 8 people.
14. \$63.
15. a. 28.
    b. 4.
    c. 0.
    d. $a^2 - 3a$.
16. a. 210.
    b. 55.
17. 420.
18. 4,845.
19. $\frac{1}{6}$.
20. $4(\frac{5}{6})^9$.
21. 95,040.
22. $2\frac{2}{3}$ ml.
23. 30 nickels,
    34 quarters.
24. Bill 6 mph,
    John 4 mph.
25. $x = -\frac{1}{3}$.
26. $x = 5$.
27. 22 and 55.
28. 1. $\{1,2,3,4,5,6,7,9\}$.
    2. $\{3,5\}$.
    3. ϕ or $\{\ \}$.
    4. $\{1,2,3,4,5,6,7,9\}$.
    5. $\{8,9,10\}$.
    6. $\{3,5,9\}$.

    b. No since $\overline{A \cup C} = \{7\}$
    but $\overline{A} \cup \overline{C} = U$.
29. a. $8i\sqrt{5}$.
    b. -10.
    c. $\frac{-2i}{3}$.
    d. $\frac{18 - 13i}{17}$.
    e. $-59 + 12i\sqrt{7}$.
30. $x = 3, y = 4$.
31. *c*.
32. *b*.
33. a. $\frac{5\sqrt{3}}{6}$.
    b. $\frac{17 + 8\sqrt{2}}{23}$.
34. $a^{10} + 30a^9b + 405a^8b^2 + 3{,}240a^7b^3$.
35. $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$.
36. $\frac{126{,}720\, x^8}{y^4}$.
37. $\frac{1}{3x}$.

38. a. $2xy^2(3x^4 + 2y)$.
b. $(x - 5)(x + 4)$.
c. $(x + 1)(x - 1)(x^2 + 1)$.
d. $(5x - 3)(x - 2)$.
e. $(x + 2)(x^2 - 2x + 4)$.
f. $(2 - x)(r + t)$.

39. a. $3xy\sqrt[3]{2y}$.
b. $2ab\sqrt[4]{a^2b^3}$.

40. a. $\dfrac{62\sqrt{3}}{3}$.
b. $28 + 10\sqrt{3}$.
c. 75.
d. 22.

41. a. $\log t = \log \pi + \frac{1}{2}(\log l - \log g)$.
b. 2.1 sec.

42. a. $a + b$.
b. $a + 2$.
c. $a - 1$.
d. $2a + b$.
e. $a + \frac{1}{2}b$.
f. $\frac{1}{3}(a + b)$.
g. $a - b$.

43. $x = 3$.

44. 20 mph.

45. $0, \frac{1}{2}, \frac{4}{3}$.

46. .22, −1.55.

47. $1 + i$, $1 - i$.

48. 10 cc.

49. Compound by \$5.09.

50. Mechanic 6 hrs., apprentice 8 hrs.

ANSWERS TO EXERCISES

Article 14, Page 24

1. $47a$.
2. $-16x$.
3. $7b$.
4. $14x^2$.
5. ab.
6. $20a$.
7. $3x^5 - 7x^4y + 3x^3y^3 - 5x^3y$.
8. $3a^2 + ab$.
9. $3x^2y + xy^2 + 3y^3$.
10. $3a^3 - 3abc + b^3$.
11. $2x^2 - 2x$.
12. $7xy - 7yz + 18zx$.
13. $20a^2b^2 + 16a^2b$.
14. $3x^3 + 10x^2y - 10xy^2$.
15. $x^2 + \frac{4}{3}xy + \frac{1}{2}y^2$.
16. $-\frac{4}{3}a^2 + \frac{7}{2}a + \frac{1}{2}$.
17. $2x^4 + x^3 - x^2 + 2x - 2$.
18. $-12a^2b + 15ab^2 - 5cd$.
19. $a + b - c$.
20. a.
21. $-4y - 4x$.
22. $2a - b - d$.
23. $2a$.
24. a.
25. $x^2 - 3xy^3 + 2$.

Article 21, Page 38

1. $12ab^2$; $6ab$; $12abc$; $-12ab$.
2. $-14xy$; $12x^2$; $\frac{5}{2}a^2x$; $-12xy^3$.
3. $8xy - 12yz + 28zx$.
4. $-3x + 3y + 3z$.
5. $9a^2b - 9ab^2$.
6. $a^2b^2c - ab^2c^2 + a^2bc^2$.
7. $15x^3y^2 - 18x^2y^3 + 24x^3y^3$.
8. $x^2 + 15x + 50$.
9. $x^2 - 25$.
10. $2x^2 + 13x - 24$.
11. $a^2x^2 - b^2y^2$.
12. $2p^2q^2 + pqr - 6r^2$.
13. $6x^3 - 19x^2 - 2x + 30$.
14. $a^2 - 2b^2 + ab + 6bc + 3ac$.
15. $a^2 + b^2 - c^2 + 2ab$.
16. $x^3 - y^3$.
17. $x^5 - 2x^4 + 10x^3 - 19x^2 + 31x - 15$.
18. $a^5 + 4ab^4$.
19. $3a$; $4b$; $-4y$.
20. $-6y$; $4z$; xz.
21. $3x^2$; $-3x$; $-5x^3$.
22. $10y^2$; $6a$; $-11c$.
23. $-8b^2x$; $7a^2b^3$; $4a^2b^4c^5$.
24. $x^2 - 3x + 1$.
25. $y^4 - 7y^3 + 4y^2$.
26. $a - b - b^2$.
27. $a - b + c$.
28. $2x^2y - 3xy^2$.
29. $-2x^2y^2 + 4x - 3y$.
30. $x + 5$.
31. $3x - 7y$.
32. $x + 2$.
33. $x - 4$.
34. $a - 6$.
35. $3x - 7$.
36. $x - 4$.
37. $\frac{2a}{b}$; $\frac{y}{ab}$; $\frac{x}{6(a + b)}$.
38. $\frac{3y}{5}$; $\frac{15c}{z}$; $x + y$.
39. 2.
40. $\frac{a}{b}$.

Article 28, Page 48

1. $a(a^2 - x)$; $x^2(x - 1)$; $2b(2a + c)$; $3a(ax + 3b - 4bx)$.
2. $2x(4 - x)$; $5x(10 + a^3x)$; $27(2 - 3y)$; $z(3z^2 - z + 1)$.
3. $19a^3x^2(2x^3 + 3a)$; $5x^3(x^2 - 2a^2 - 3a^3)$.
4. $(x - a)(x + b)$.
5. $(3x + 2b)(2x - 3a)$.
6. $(4a - x^2)(3a + b)$.
7. $(x^2 + 1)(3x - 5)$.
8. $(x + 8)(x + 3)$.
9. $(x - 6)(x - 4)$.
10. $(y - 9)(y - 9)$.
11. $(z + 7)(z + 5)$
12. $(x - 2)(x + 1)$.
13. $(x + 2)(x - 1)$.
14. $(y + 23)(y - 5)$.
15. $(x^2 - 17)(x^2 + 3)$.
16. $(2x + 1)(x + 1)$.
17. $(3x + 2)(x + 2)$.
18. $(3y - 1)(2y + 3)$.

Article 33, Page 59

1. $4x^2$; $27a^3$; 16[illegible]y^2; $64a^3b^3c^3$; $16x^4y^4z^4$.
2. $y^2 - 2yz + z^2$; $z^2 - 2zy + y^2$; $4x^2 + 4ax + a^2$; $a^2 - 6ab + 9b^2$.
3. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$; $x^2 + y^2 + z^2 - 2xy + 2yz - 2zx$; $a^2 + 4b^2 + 9c^2 - 4ab - 12bc + 6ca$.
4. $x^3 + y^3 + z^3 + 3(x^2y + y^2x) + 3(y^2z + z^2y) + 3(z^2x + x^2z) + 6xyz$.
5. $a^3 - 8b^3 + 27c^3 - 6a^2b + 12ab^2 + 36b^2c - 54bc^2 + 27c^2a + 9ca^2 - 36abc$.
6. $27x^3 + 8y^3 + z^3 + 54x^2y + 36xy^2 + 12y^2z + 6yz^2 + 9z^2x + 27zx^2 + 36xyz$.
7. $6a$; $4x^2y^2$; $3ax$; $3bz^2$.
8. $2(x - a)$; $2(a + b)$; $2x(2x - 3y)$.
9. $(a + b)(c - d)^2$. 10. $x - 5y$. 11. $2x^2 - 3x + 5$. 12. $ab + 3bc - 2ca$.

Article 38, Page 63

1. $6a^3b^3$; $-28x^4y^7z$; bx.
2. $2a^2b^2$; $-2x^2y^2z^2$; $3xy^{1/3}$.
3. $16x^6$; $4a^2b$; x^2y^4; ac^2.
4. $\dfrac{2}{x^{1/4}}$; $\dfrac{3}{a^{2/3}}$; $\dfrac{1}{5x^{1/2}}$; $\dfrac{4a^3}{x^2}$; $3a^2$; $\dfrac{1}{4a^2}$.
5. $\dfrac{3c^4x^2}{5a^3y^2}$; $\dfrac{x^{2a}}{y^{2b}}$; $\dfrac{6}{x^{1/2}}$; $\dfrac{1}{3a^2x^2}$.

Article 43, Page 71

1. $\sqrt[5]{x^3}$; $\dfrac{1}{\sqrt{a}}$; $\dfrac{5}{\sqrt{x}}$; $2\sqrt[4]{b}$; $\dfrac{1}{2\sqrt[3]{c}}$; $\dfrac{2}{\sqrt[x]{a}}$.
2. $\dfrac{2}{\sqrt[6]{a^5}}$; $\dfrac{\sqrt{a}}{2\sqrt[3]{x^2}}$; $\dfrac{21}{\sqrt{a^3}}$; $\dfrac{2}{\sqrt{a}}$.
3. $a^{13/6}$; 1; $a^{\frac{n}{6}}$.
4. 8; $\frac{1}{32}$; 25; $\frac{1}{4}$; $\frac{1}{216}$; 9.
5. $12\sqrt{2}$; $7\sqrt{3}$; $5\sqrt[3]{2}$; $6\sqrt[3]{2}$; $24\sqrt{5}$; $6a\sqrt{a}$.
6. $14\sqrt{6}$; $12\sqrt{3}$; $14\sqrt[3]{9}$; $288\sqrt{2}$.
7. $\sqrt{5}$; 6; $42\sqrt{2}$; $12\sqrt{6}$. 8. $14\sqrt{5}$. 9. $\sqrt{7}$. 10. $-15\sqrt{3}$.

Article 50, Page 79

1. $5i\sqrt{2}$; $18i\sqrt{3}$; $20i - 6$.
2. $6i^2 = -6$; $(2 + \sqrt{6}) - i(2\sqrt{3} - \sqrt{2})$; $\sqrt{15} + 2\sqrt{10} - 2\sqrt{6} - 3$; $20 + 21i$.
3. $\frac{3}{i} = -3i$; $-\frac{2}{\sqrt{2} + \sqrt{3}} = -2(\sqrt{2} - \sqrt{3})$; $\frac{a + \sqrt{2}}{2 - \sqrt{2}}$; $\frac{(2\sqrt{5} - 3\sqrt{3}) + (\sqrt{15} - 6)i}{7} = -.103 - 1.41i$.
4. $\frac{21}{13} + i\frac{1}{13}$; i; $\frac{4 - 3\sqrt{6}i}{5}$.
5. $-7 + 22i$; $12 - 4.625i$; $-2 + 11i$.
6. $|A| = 10$; $|B| = 13$; $|C| = 5.831$.
7. 1.614; 1; 0.5976.
8. $.006i - .276$.
9. $-5 - 40i = -5(1 + 8i)$.
10. $\frac{5i}{4}$.

Article 54, Page 88

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.
2. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.
3. $64a^6 + 576a^5 + 2160a^4 + 4320a^3 + 4860a^2 + 2916a + 729$.
4. $243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$.
5. $c^8 + 4c^6d^2 + 6c^4d^4 + 4c^2d^6 + d^8$.
6. $\frac{x^8}{16} + \frac{x^6y^3}{2} + \frac{3x^4y^6}{2} + 2x^2y^9 + y^{12}$.
7. $1 + x + x^2 + x^3 + x^4 + \ldots.$
8. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots.$
9. $1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}x^4 + \ldots.$
10. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots.$
11. $1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} - \ldots.$
12. $\sqrt[3]{a} + \frac{b}{3\sqrt[3]{a^2}} - \frac{b^2}{9\sqrt[3]{a^5}} + \frac{5b^3}{81\sqrt[3]{a^8}} - \frac{10b^4}{243\sqrt[3]{a^{11}}} + \ldots.$

Article 61, Page 107

1. 5.
2. 4.
3. 7.
4. 3.
5. 5.
6. 5.
7. 1.
8. 3.
9. $1\frac{5}{16}$.
10. 4.
11. $24\frac{17}{19}$.
12. 10.
13. 7.
14. $-2\frac{5}{24}$.
15. $23\frac{8}{19}$.

Article 65, Page 117

1. $x = 2, y = 1.$
2. $x = 3, y = 5.$
3. $x = 7, y = 8.$
4. $u = 4, v = -1.$
5. $x = 12, y = 8.$
6. $x = 10, y = 6.$
7. $y = 35, z = 45.$
8. $x = 1, y = 2, z = 3.$
9. $x = -2, y = 4, z = 1.$
10. $x = 2, y = -2, z = 5.$
11. $x = 2, y = 3, z = 1.$
12. $x = 4, y = -3, z = 2.$
13. $x = 1, y = 2, z = 3.$
14. $x = 5, y = 1, z = 3, u = 2.$
15. $u = 6, v = -7, w = 8, x = -9.$

Article 73, Page 134

1. $1\frac{1}{6}; -1.$
2. $1\frac{3}{4}; 1\frac{1}{2}.$
3. $-3; \frac{7}{11}.$
4. $-4; -1\frac{1}{5}.$
5. $-2\frac{1}{3}b; 1\frac{2}{3}b.$
6. $\frac{5}{12}; -\frac{3}{8}.$
7. $2\frac{1}{5}; -5.$
8. $11; 3\frac{2}{3}.$
9. $2; -1\frac{5}{6}.$
10. $2; \frac{1}{3}.$
11. $4; 1\frac{1}{3}.$
12. $7; -1\frac{1}{3}.$
13. $1\frac{2}{3}; -3.$
14. $1\frac{1}{2}; -5.$
15. $-4; -\frac{1}{5}.$
16. $7; -2\frac{1}{2}.$
17. $\dfrac{1 \pm \sqrt{13}}{6}.$
18. $\dfrac{17 \pm \sqrt{89}}{10}.$

Article 80, Page 141

1. 1. $x_1 = 4; x_2 = -2 + i\sqrt{3}; x_3 = -2 - i\sqrt{3}.$
2. $x_1 = \sqrt[3]{2(2+\sqrt{6})} + \sqrt[3]{2(2-\sqrt{6})} = .1074;$
   $x_2 = -1.5536 + 2.620i; x_3 = -1.5536 - 2.620i.$
3. $1.6335; -.8168 + .5012i; -.8168 - .5012i.$
4. $-.3896; 1.6948 \pm 1.7650i.$
5. $2.1958; -1.0979 \pm .7849i.$
6. $6, -3, -3.$
7. $1, 3, 4 (D < 0).$
8. $-7, 3, 3.$
9. $2, 2, -1.$
10. $-1, -5, 6 (D < 0).$
11. $x = 1, -1, -4 \pm \sqrt{6}.$
12. $x = -\frac{1}{2}, -3, 2, 1\frac{1}{2}.$

Article 84, Page 158

1. 12; 17.
2. 16.
3. 17; 18.
4. \$10,800; \$3600; \$1800.
5. 60; 20; 15 yrs.
6. 25; 17.
7. 20.
8. 20.
9. 70¢.
10. 75.
11. 121 sq. ft.
12. 12 × 40; 16 × 30 yds.
13. 860.
14. 8 hrs.
15. 4, 6 oz.
16. 3 ft.
17. 48; 110 ft.
18. 3 games.
19. 4%.
20. 3 mi./hr.

Article 92, Page 169

1. 4. 2. 4. 3. 7. 4. 1.716. 5. 4. 7. 24. 8. 7.325. 9. 0.663.

Article 103, Page 189

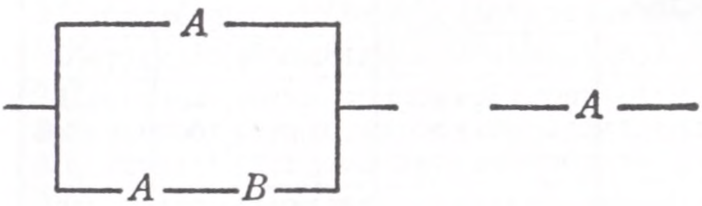
1. 36,460.
2. 82.26.
3. 1.2×10^{12}.
4. 2.213.
5. .1151.
6. 45,510.
7. 2.104.
8. 16.38.
9. .4255.
10. 77.72.
11. 1.393.
12. 4.757.
13. 9.472.
14. 2.65×10^7.
15. 8.218.
16. 130.
17. 1.571×10^7.
18. 199.7.
19. 1.817.
20. 2659.

Article 106, Page 198

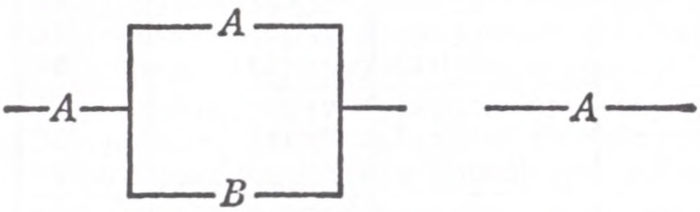
1. 2.243.
2. $-.5303$.
3. $-.491$.
4. 3.854.
5. $326,400.
6. $4153.
7. $18\frac{1}{2}$ yrs.
8. 8.16%.
9. $267.60.
10. $0.029 = 2.9¢ more.

Article 141, Page 260

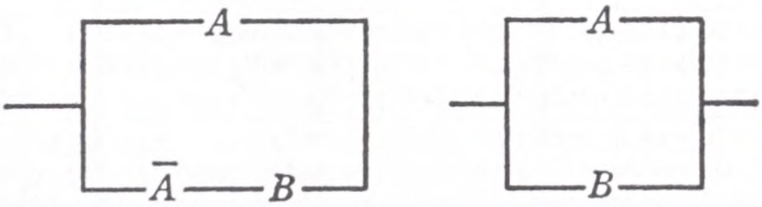
1. $A + BC$.
2. A.



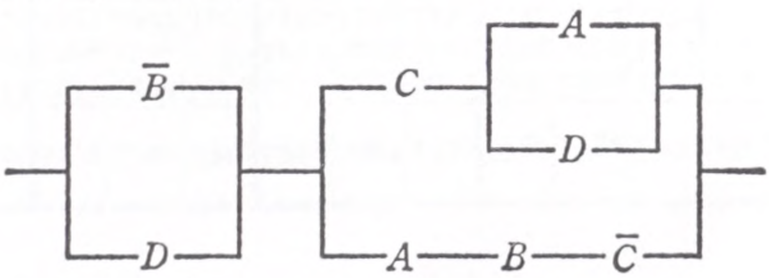
3. A.



4. $A \vee B$.



5. $AC \vee A(B \vee C)$.
6. $(A \vee B)(AB \vee C)$; $(A \vee B)C$.
7. $(B \vee C)(B \vee C) \vee ABC$.
8. $(B \vee D)[C(A \vee D) \vee ABC]$.



9. Vector 7,7.
10. Vector 9, −3.
11. Vector 14,10 of magnitude 17.2 . . . .
12. 5; 6.32 . . . ; 7.8 . . . ; 8.06 . . . .
13. 7.07 . . . miles per hour north.
14. 46.
15. −3.
16. 75.
17. Vector of magnitude 52 directed downward perpendicular to the plane of the paper.
18. Vector of magnitude 69 directed upward perpendicular to the plane of the paper.
19. Vector 7,9,2.
20. Vector 2,11,−5.
21. Vector 4,6,−3; magnitude 7.81 . . . .
22. −5.
23. −96.
24. 79 foot-pounds.
25. Vector 7,4,−2.
26. Vector 9,−27,0. The zero value of the *Z*-coordinate indicates that the vector lies in the *XY*-plane.
27. Vector 22,1,−10.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374									
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
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48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
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51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
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58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
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61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
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75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
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83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
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90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
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